TIME IN ASTRONOMY, ROTATION OF THE EARTH - THEORY AND OBSERVATIONS

Jan Vondrák, Astronomical Institute Prague

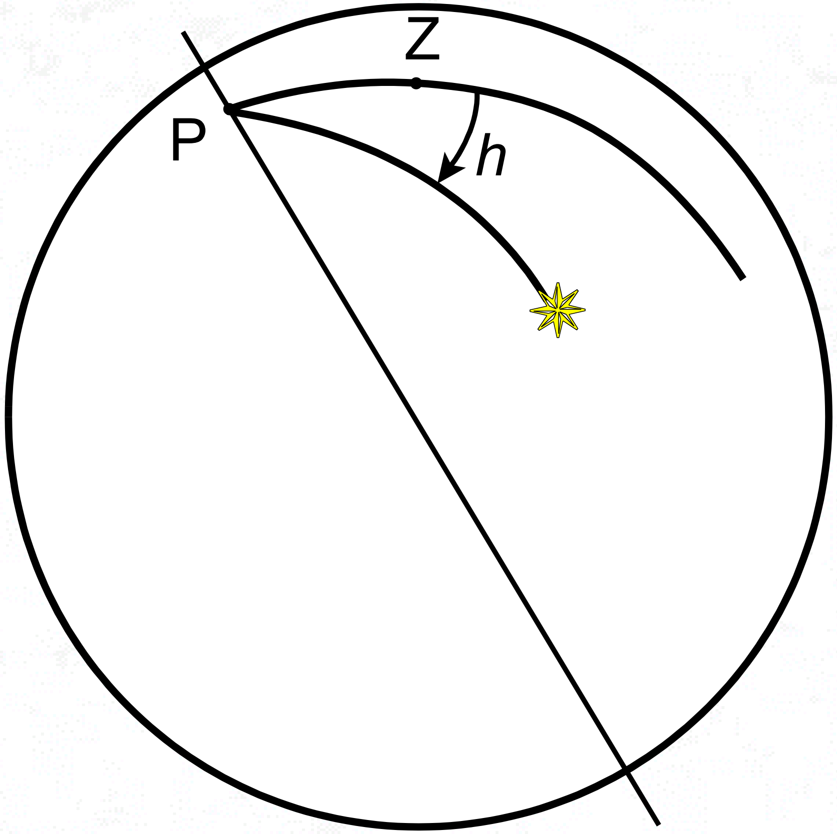
- **PART 1:**
  - **Time in astronomy, its measurement and techniques;**
  - **Theory of rotation of the Earth:**
    - **rigid;**
    - **non-rigid.**

HOW TIME IS MEASURED IN ASTRONOMY

Historical overview of the methods, most recent results

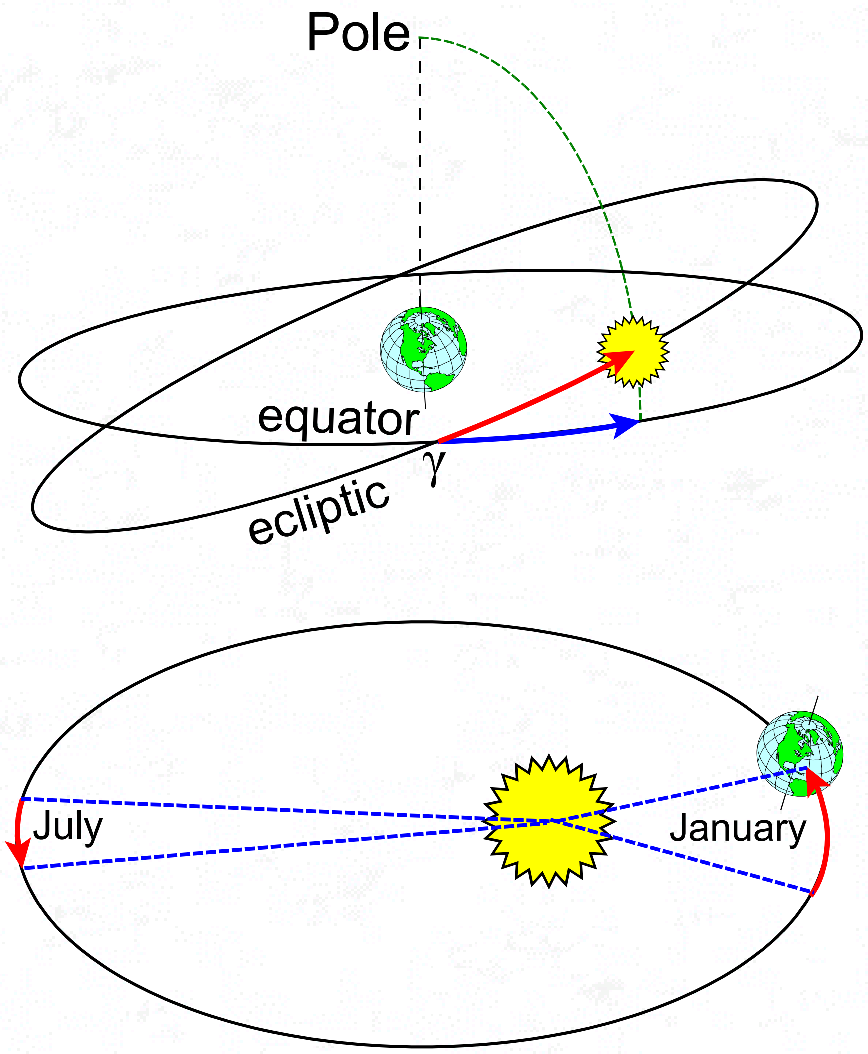
- **Time measurement was, from time immemorial, closely linked with astrometric observations. Generally holds:**
  - **time is measured by means of periodically repeating phenomena with sufficient stability (rotation of celestial bodies, their orbital motion, oscillation of pendulums, crystals, atoms...)**
  - **for many centuries the basis of time measurement was the rotation of the Earth (apparent daily motion of celestial bodies on the sky),**
  - **for longer time intervals serves the motion of the Moon and Sun (calendar):**
    - **changes of lunar phases (week, month),**
    - **changes of seasons (year),**

- for shorter time intervals clocks were used, e.g.:
  - water clocks,
  - mechanical clocks (with different regulators - foliot, pendulum, balance wheel, ...),
  - crystal clocks,
  - atomic clocks.
- Historically the oldest astronomical measurement of time - observation of the Sun (true/mean solar time): gnomons, sundials
- True solar time = $h_☉$ + 12h (it is not uniform!)

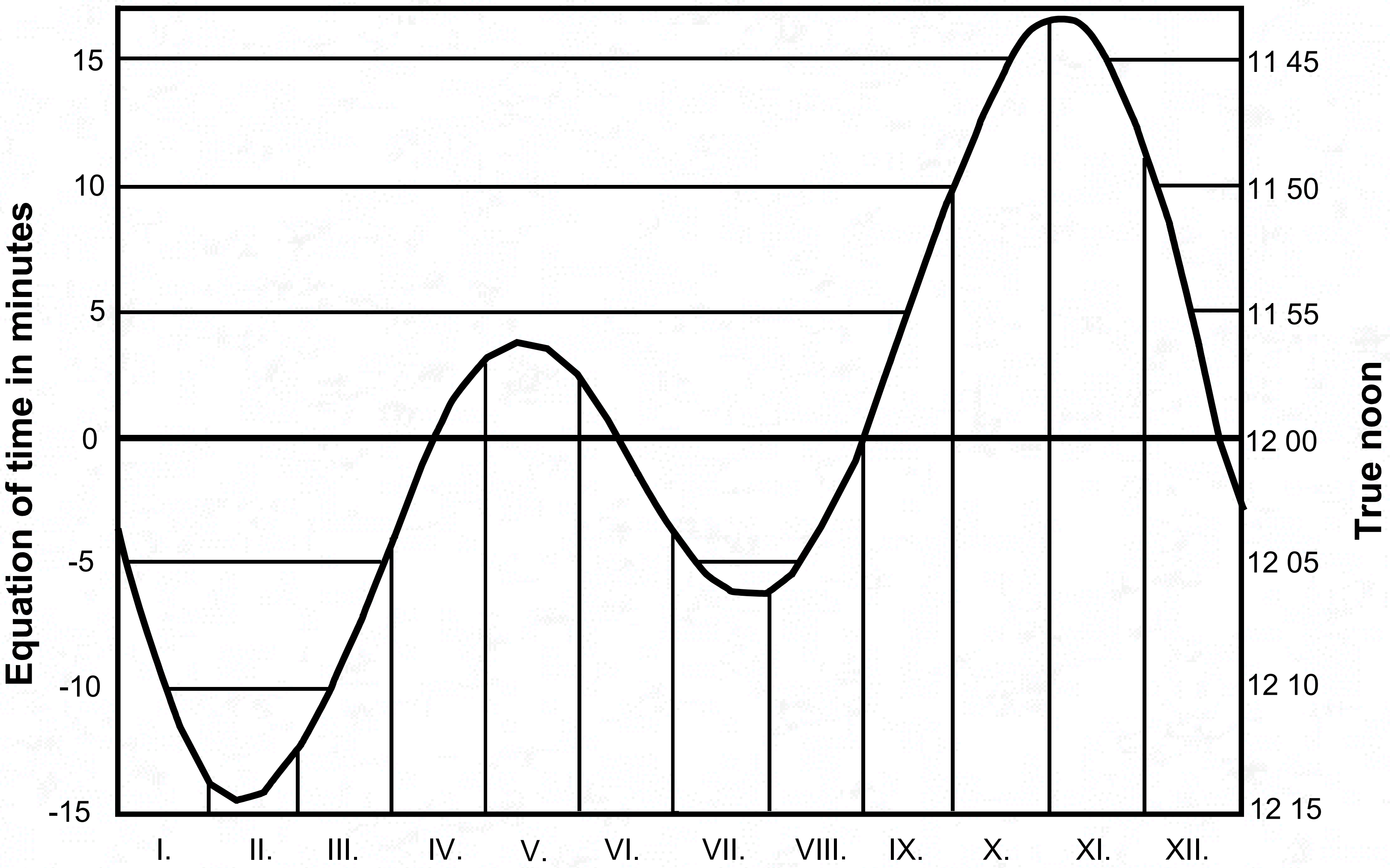


Gnomon at Purple Mountain Obs., 1439

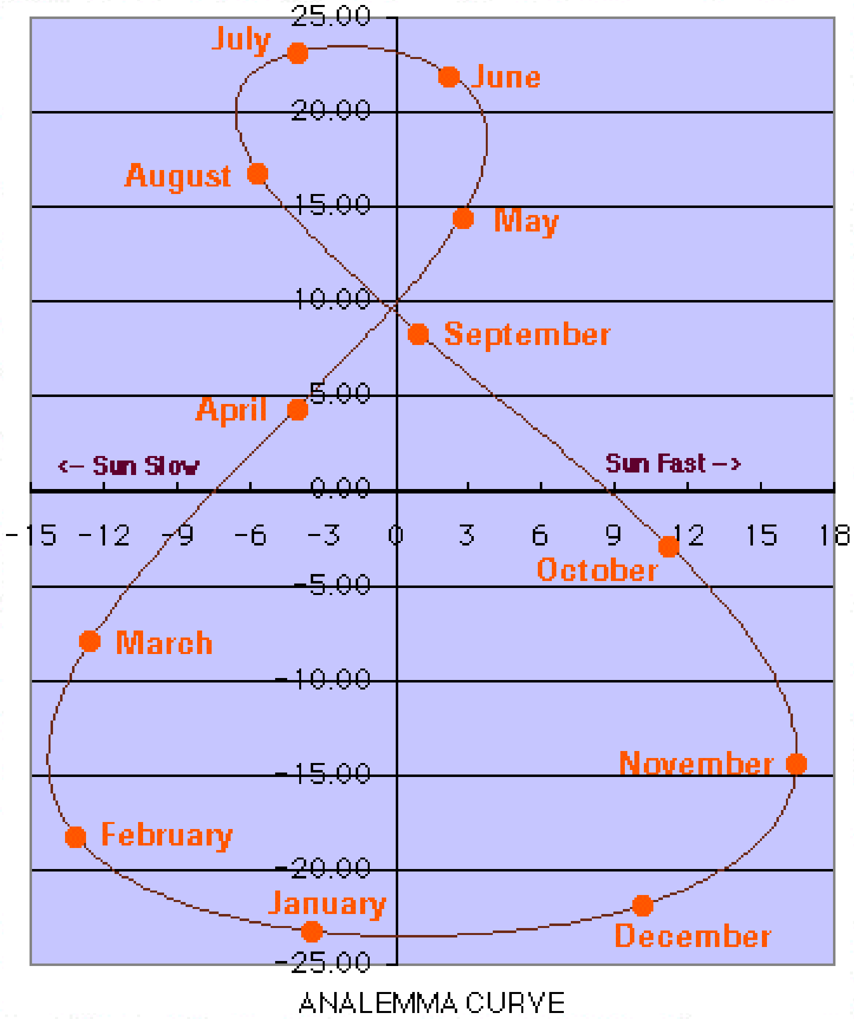
- **Difference between true and mean solar time (equation of time):**
  - ♦ **Orbital motion of the Earth around the Sun along the ecliptic; projection on the plane of equator causes the semiannual variation with the amplitude of about 9.9 minutes and zero values at equinoxes and solstices.**
  - ♦ **Elliptical motion of the Earth around the Sun with non-uniform velocity results in annual variation with the amplitude of about 7.6 minutes and zero values at transits through perihelium and aphelium.**



Equation of time:

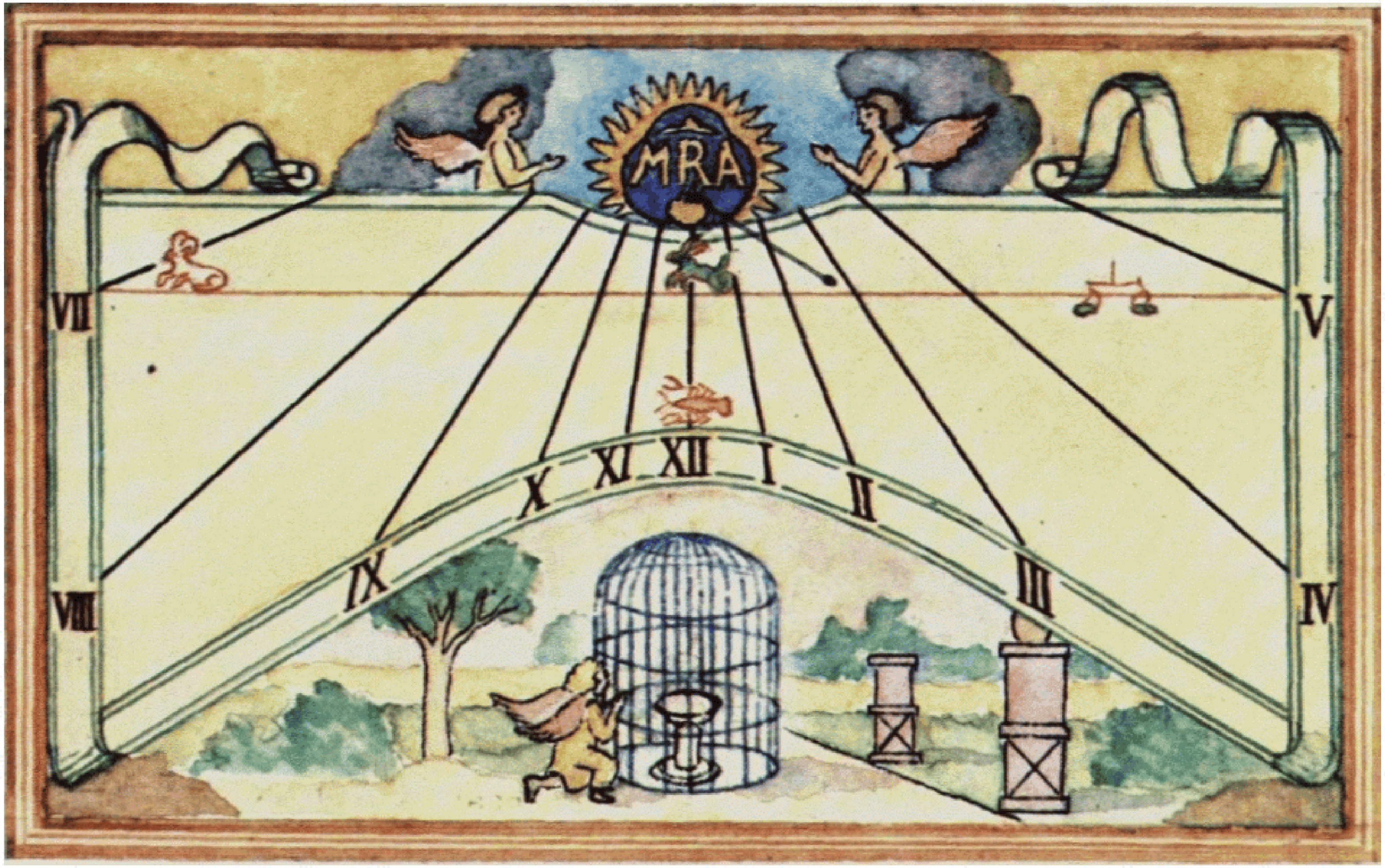


Sundials often use **analemma** to correct true to mean solar time:

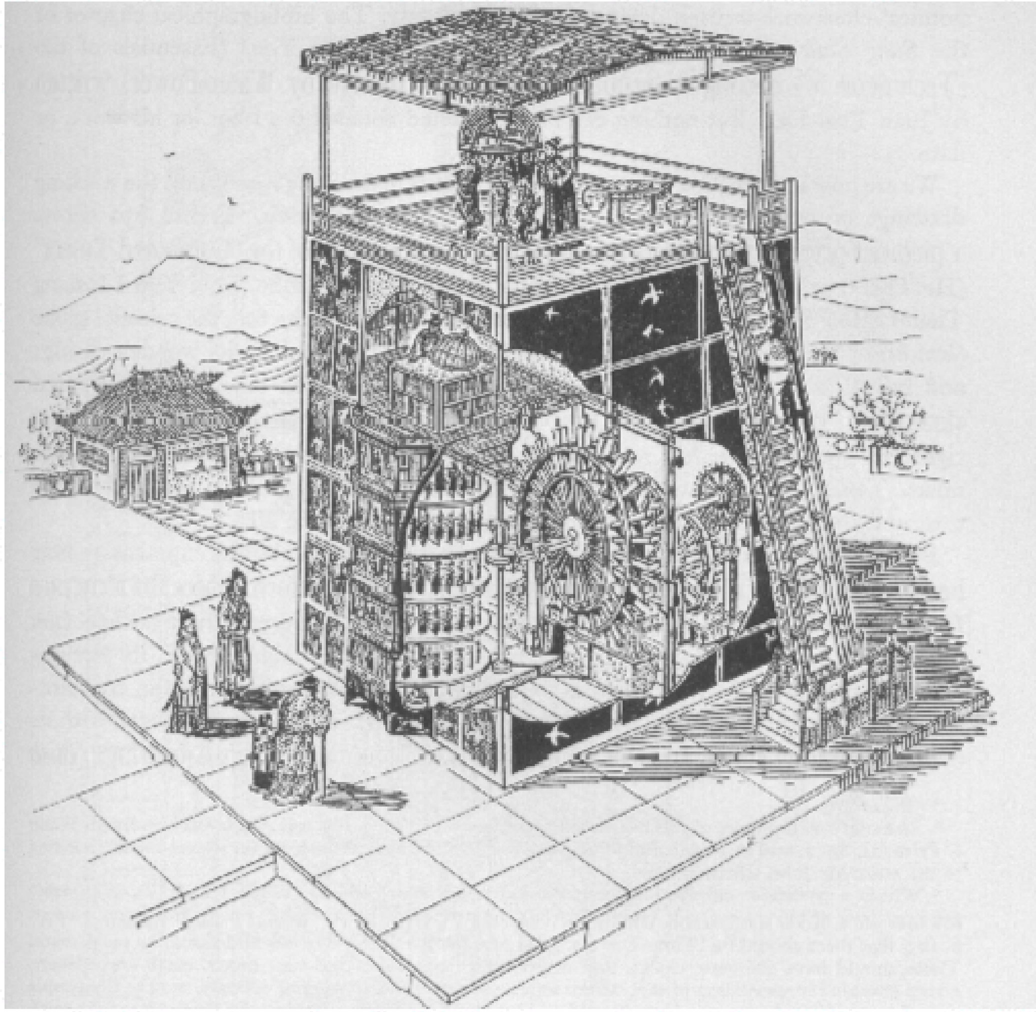


Sundial in Imperial palace, Beijing

Sundial, Klementinum (Prague)

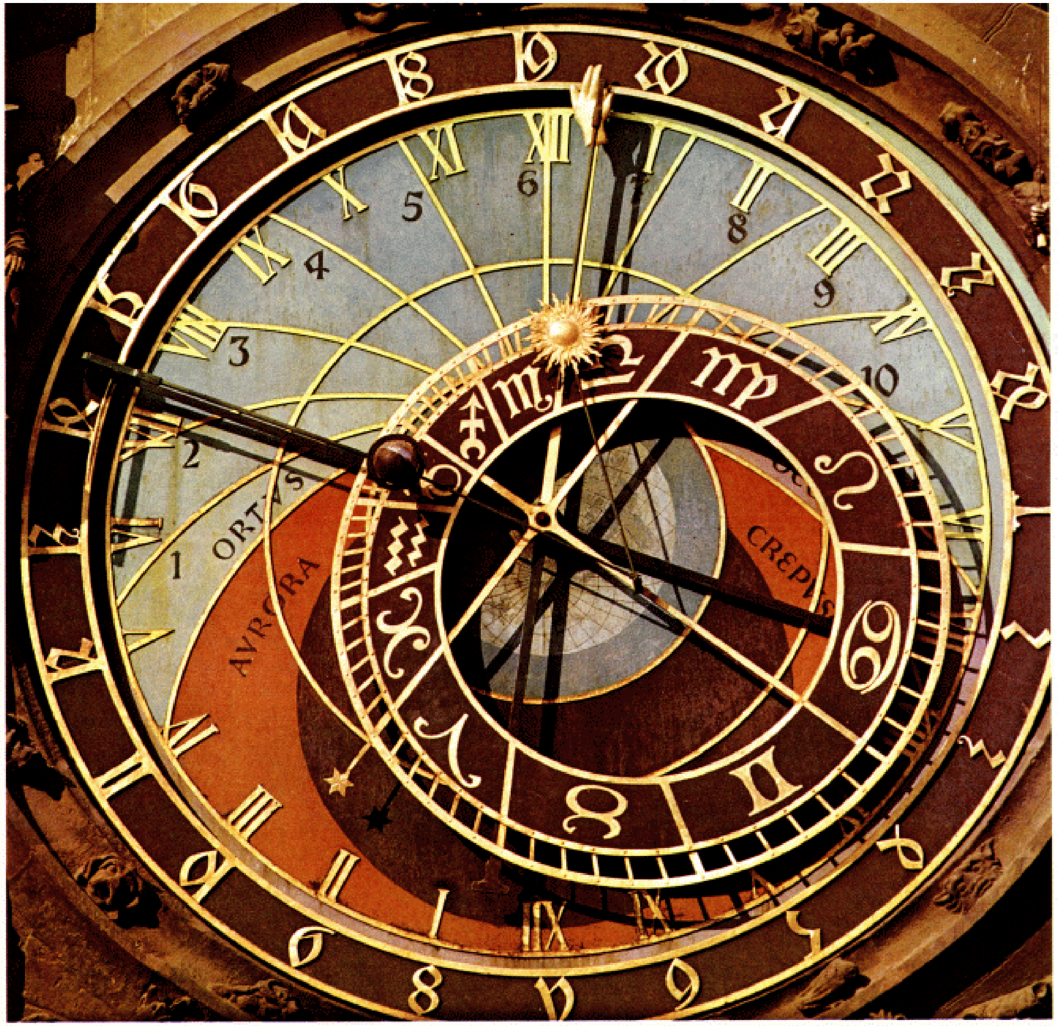


Su Sung's astronomical clock, China, app. 1088

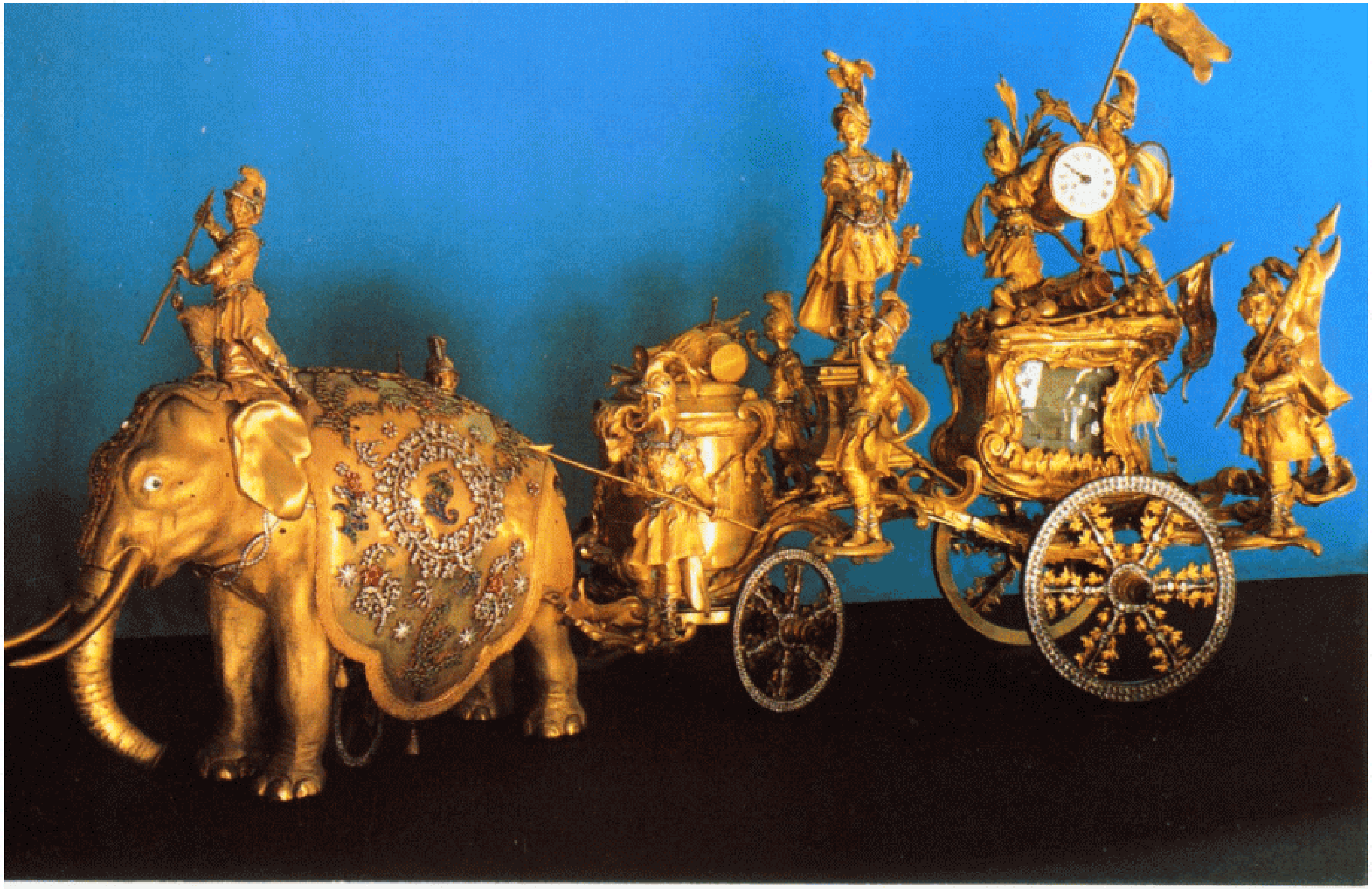


Water clock, Imperial palace, Beijing

Prague astronomical clock, 15th century



English clock of 18th cent. - Imp. palace, Beijing



- Local time (directly measured astronomically) is different on different meridians; in the past, each larger township used its own local time.
- Problems with transfer of time to another location (e.g., when determining geographic longitude, especially in navigation on the sea).
- Development of railway transport forced (and invention of telegraph enabled) unification of time and international adoption of zero (Greenwich) meridian and zonal times (Washington, 1884) ⇒ Greenwich Mean Time (GMT), later renamed to Universal Time (UT).
- Interesting fact: until 1925 time in astronomy was reckoned from the noon, not midnight as today!

- More precise astronomical measurement of time is given by observing the stars; sidereal time is defined as hour angle of the equinox: $S = h + \alpha$ (local), $S_G = h + \alpha - \lambda$ (Greenwich), and from it, by using a conventional relation, Universal Time (UT).
- Greenwich sidereal time at midnight UT (in seconds, T in centuries from the epoch J2000,0):

$$S_G^0 = 24110.54841 + 8640184.812866T + 0.093104T^2 - 6.2\times10^{-6}T^3 + \Delta\psi\cos\varepsilon$$

- Note: Both sidereal and Universal time is the angle given by Earth rotation, measured in time units!

Mean sid. time

equation of equinoxes

- **New relation to recalculate stellar angle $\theta = h + \alpha$ (right ascension reckoned from CIO) and Universal Time UT1:**

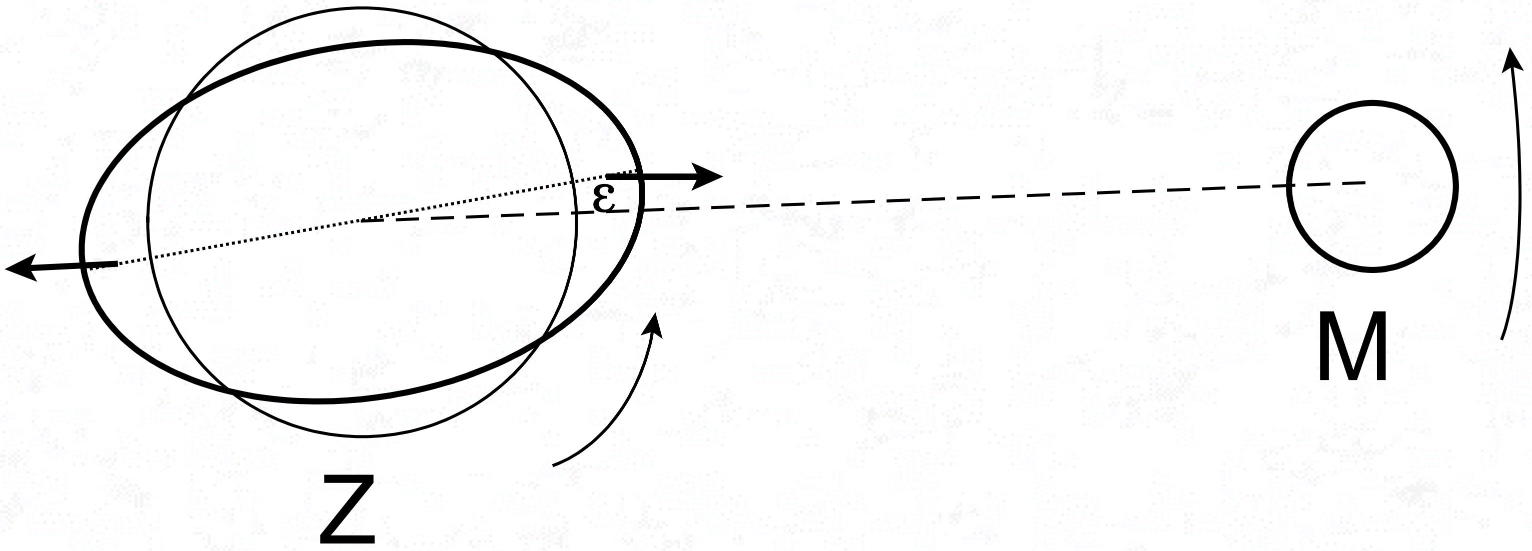
$$\theta(T_u) = 2\pi(0.7790572732640 + 1.00273781191135448T_u), \quad \text{where}$$
$$T_u = JD(\mathrm{UT1}) - 2451545.0$$

"Family" of universal times:

- UT0 - directly astronomically measured (not corrected) time; it depends on the observer's position.
- UT1 - UT0 corrected for the change of longitude due to polar motion:
  - $UT1=UT0+1/15(x\sin\lambda + y\cos\lambda)\tan\varphi$
- UT2 - preliminary uniform time scale, coming from UT1 by correcting it by seasonal variations of the Earth's speed of rotation (nowadays not used any more):
  - $UT2=UT1+0.020\sin 2\pi t-0.012\cos 2\pi t-0.006\sin 4\pi t+0.007\cos 4\pi t$
- UTC - coordinated universal time; it is transmitted as time signal, its unit is SI second, its difference from TAI changed stepwise by whole number of seconds so that
  - $|UTC-UT1|<0.9s$

Transit instrument Zeiss 100-1000mm, GO Pecný

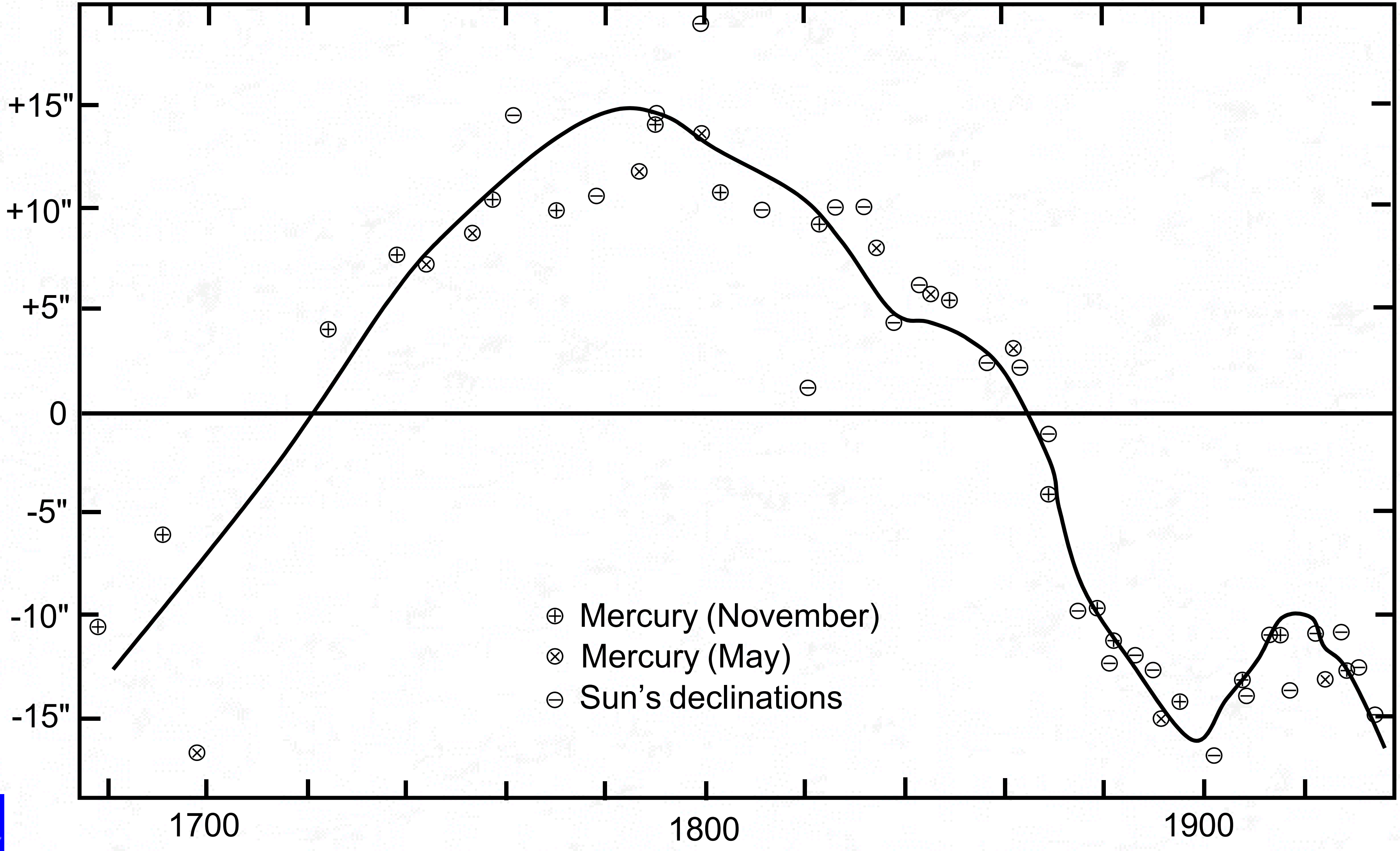
- **Rotating Earth cannot be used as an ideal clock:**
  - ♦ **Variable speed of rotation**
    - **Secular deceleration of rotation due to tidal friction (noticed already by Halley in 1695 in orbital motion of the Moon, later, in 18th century, studied by Laplace, its influence on Earth's rotation was implied at the end of 19th century by G. Darwin.)**



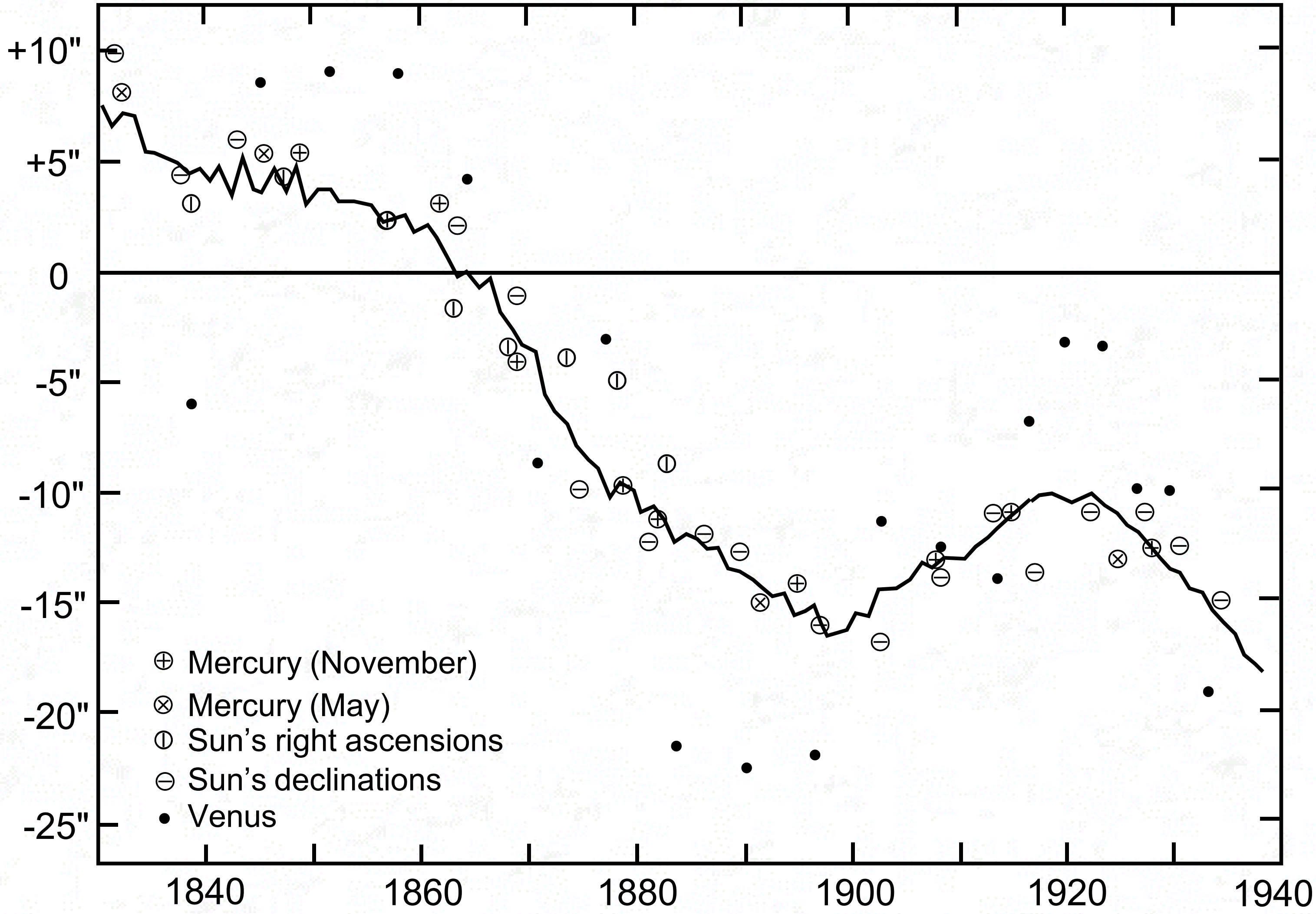
- **Tidal friction decelerates the speed of Earth's rotation (length of day increases by about 0,002s per century), and accelerates the Moon in its orbit (its ecliptical longitude decreases by about 11" per century^2 with respect to the theory) so that the total angular momentum of the system Earth-Moon is conserved.**
- **As a consequence of the third Kepler's law (n^2a^3=const.) the Moon moves away from the Earth by about 3cm per year.**

- Irregular decade variations of the Earth's speed of rotation (probably due to the changes on the core-mantle boundary?). De Sitter (1927) and Spencer Jones (1932, 1939) found, from the observations of Mercury, Venus, Sun and Moon, that the deviations of their ecliptic longitudes from theory can be expressed as
  - $\Delta L = a + bt + ct^2 + B(t)$, where
  - coefficients *c* are directly proportional to the mean motions for Mercury, Venus and Sun,
  - irregular variations *B* are directly proportional to the mean motions for all observed bodies.

Fluctuations in lunar longitude from 1680, based on observations of the Moon, Sun and Mercury (after Spencer-Jones, 1939).



Fluctuations of lunar longitude from 1830, calculated from observations of the Moon, Sun, Mercury and Venus (after Spencer-Jones, 1939)



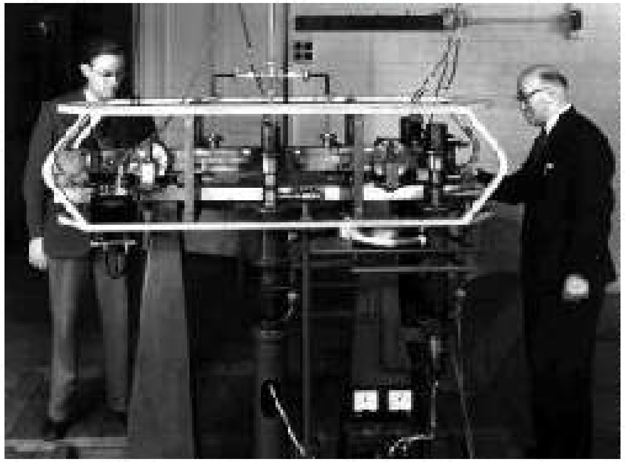
- **Conclusion:** **All these changes have just one common cause - irregularity of the time scale used (UT). Secular term *c* is caused by a slow deceleration of the rotation due to tidal friction, irregular fluctuations *B* by changes of the moment of inertia of the Earth; different *c* of the Moon is caused by its tidal acceleration in orbit.**

- Seasonal variations of Earth's speed of rotation (due to zonal winds and tidal deformations of the Earth) were discovered by Stoyko (1937). These are equal, in average, to the difference UT2-UT1.

⇓

- Clemence (1948) proposed to adopt a new time scale, **Newtonian time**, as independent variable in the equations of motion of celestial bodies. IAU in Rome (1952) accepted this proposal under the name **Ephemeris Time (ET)**.
  - ♦ Origin: 0. January 1900 12h ET, when geometric mean longitude of the Sun was equal to 279°41′48.04″ (after Newcomb's theory of Earth's motion).
  - ♦ Unit: 1s=1/31556925.9747 of tropical year 1900 (corresponding to the unit of rotational time UT1 around 1830).

- **Practical realization from the observation of occultations of the stars by the Moon (attention: other body than the definition, ET depends on the theory of lunar motion: j=0, j=1, j=2), and is known only after the observations are elaborated.**
- **In 1955 first atomic clock was constructed. BIPM in 1967 adopted a new definition of time; the unit is a second of SI, realized by means of atomic clocks: (9192631770 periods of the radiation corresponding to the transition between two hyperfine levels of the ground state of the atom of C-133). It is selected so that the SI second = ET second.**
- **Since 1971 International Atomic Time (TAI) is created whose origin has been identified with 0h UT2, January 1, 1958.**



- **Thus the creation and determination of time was taken from astronomers and given to physicist;**
- **Astronomical observations are further used to determine “time” UT1 in relation to TAI, as a measure of variations in the speed of rotation of the Earth;**
- **However, in future the situation might reverse: millisecond pulsars have a very stable rotation!**

"Family" of proper and coordinate time scales (adopted by IAU in 1991; in agreement with the theory of general relativity the time depends on the velocity and intensity of gravitational field):

◆ Terrestrial Time (TT) is proper time. It is the argument of apparent geocentric ephemerides. Its unit is a second SI on the geoid, and the origin is given by the difference TT-TAI=32.184s at the instant January 1, 1977 at 0h. It is the continuation of Ephemeris Time.

◆ Geocentric Coordinate Time(TCG):

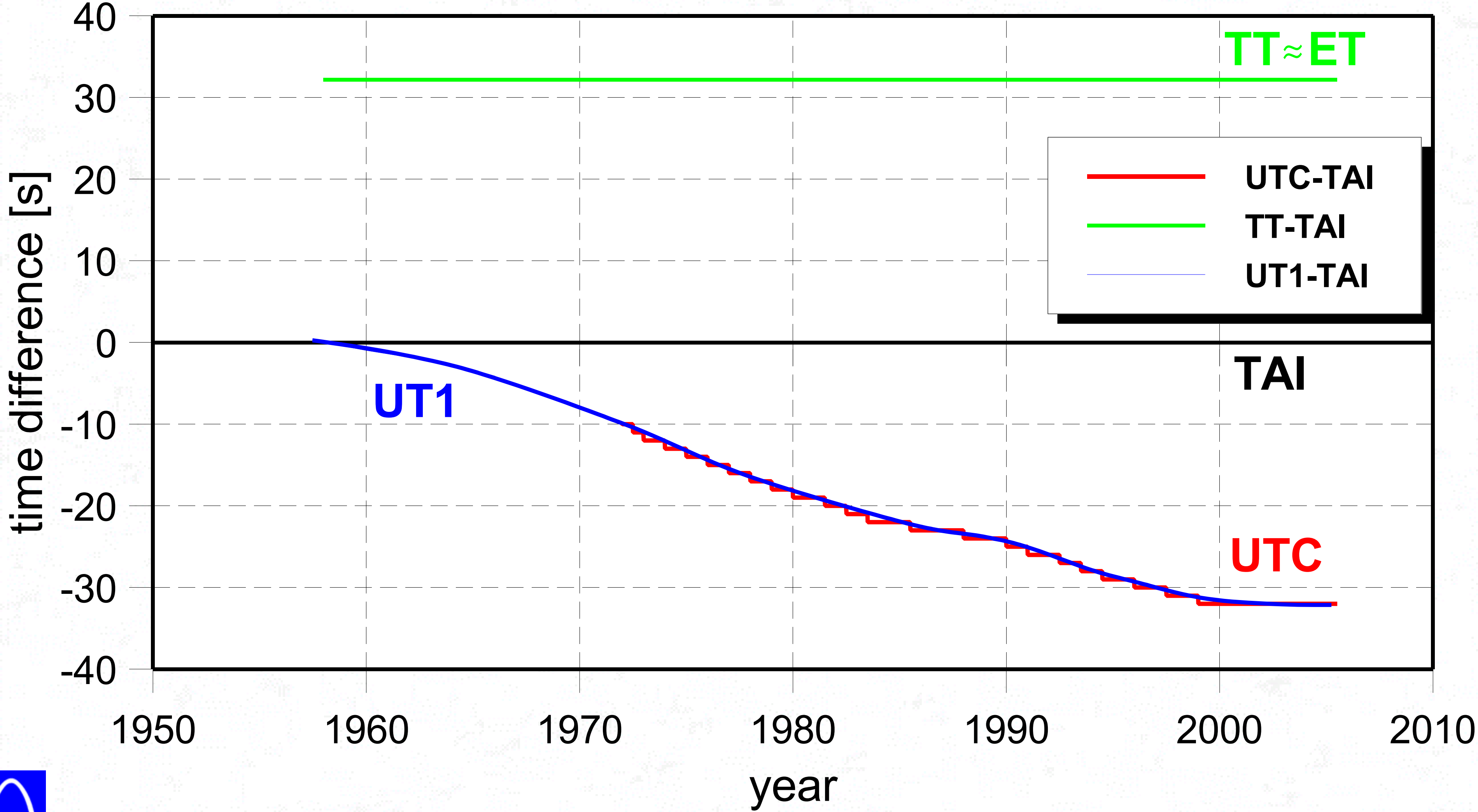
$$\mathrm{TCG} = \mathrm{TT} + 60.214\,(\mathrm{JD} - 2443144.5)\ [\mu s]$$

◆ Barycentric Coordinate Time (TCB):

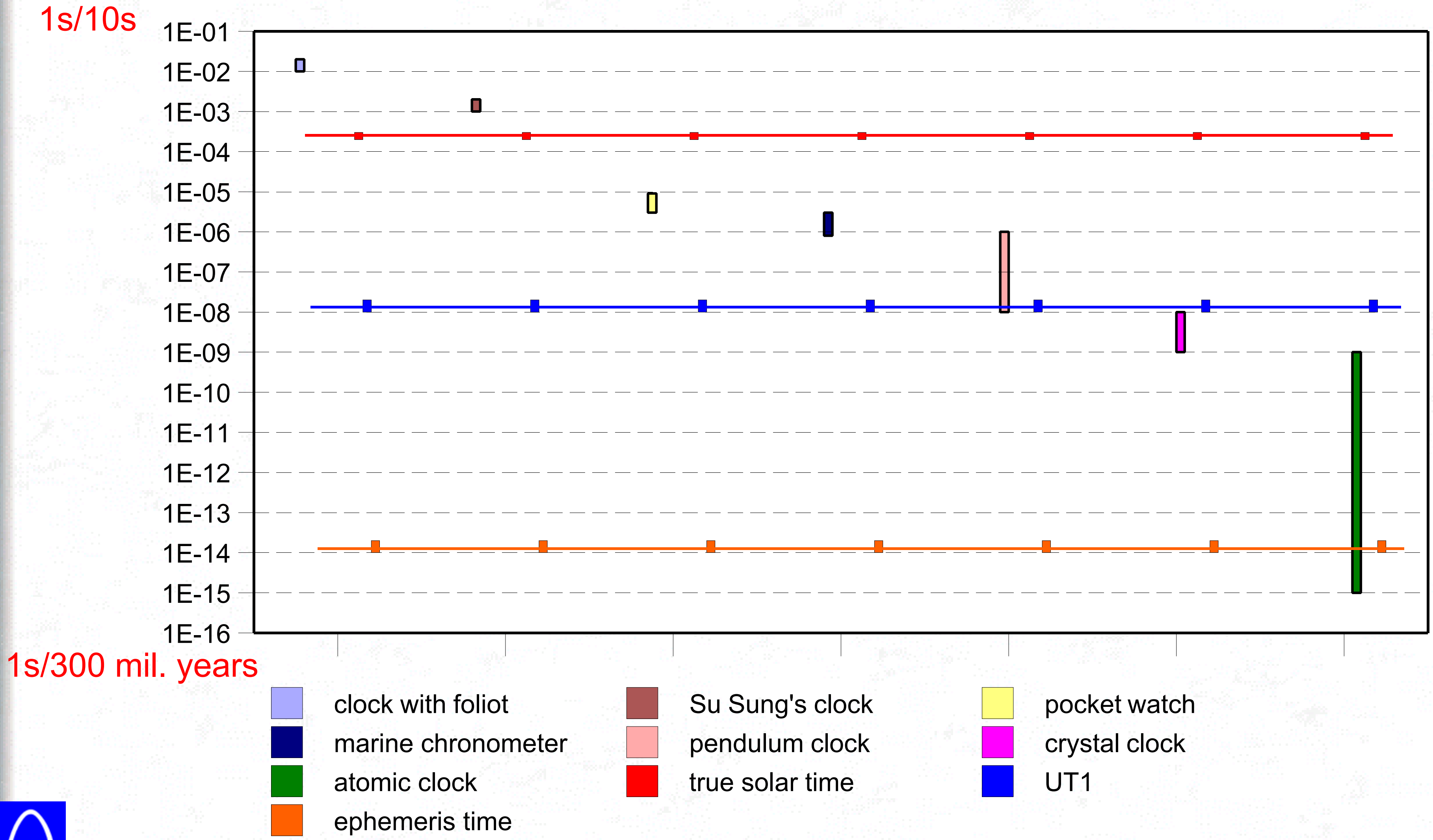
$$\mathrm{TCB} = \mathrm{TT} + 1339.636\,(\mathrm{JD} - 2443144.5) + 1658 \sin g + 14 \sin 2g + \dots\ [\mu s]$$

where $g = 357.53 + 0.9856003\,(\mathrm{JD} - 2451545.0)\ [^\circ]$

Relation of time scales TT, UT1, UTC, and TAI



Relative stability of time realization



International centers for evaluation of observations and worldwide coordination of time:

- **BIH - Bureau International de l'Heure, Paris (1911-1987); originally created for coordination of emitting time signals in UT, later on (after introducing atomic time) for producing TAI and measuring the variations of Earth's rotation.**
- **BIPM - Bureau International des Poids et Mesures, Sèvres; since 1985 responsible for producing TAI.**
- **IERS - International Earth Rotation and Reference Systems Service, CB in Paris (since 1988), Frankfurt (since 2000); created as a common service of IAU/IUGG to monitor all components of Earth orientation, and to create and maintain terrestrial and celestial reference systems.**

THEORY OF EARTH'S ROTATION

- Earth rotation in a wider sense - global orientation of the body (precession-nutation, polar motion, proper rotation), it is influenced by:
  - ♦ external forces (torques by the Moon, Sun, planets);
  - ♦ internal forces (internal composition of the Earth, transfer of mass in the atmosphere, hydrosphere, boundary between the mantle and the core ...).
- It has a fundamental importance in astronomy, especially for transformation between celestial and terrestrial reference systems, but also further applications in related specializations:
  - space navigation;
  - geophysics;
  - geodesy etc...

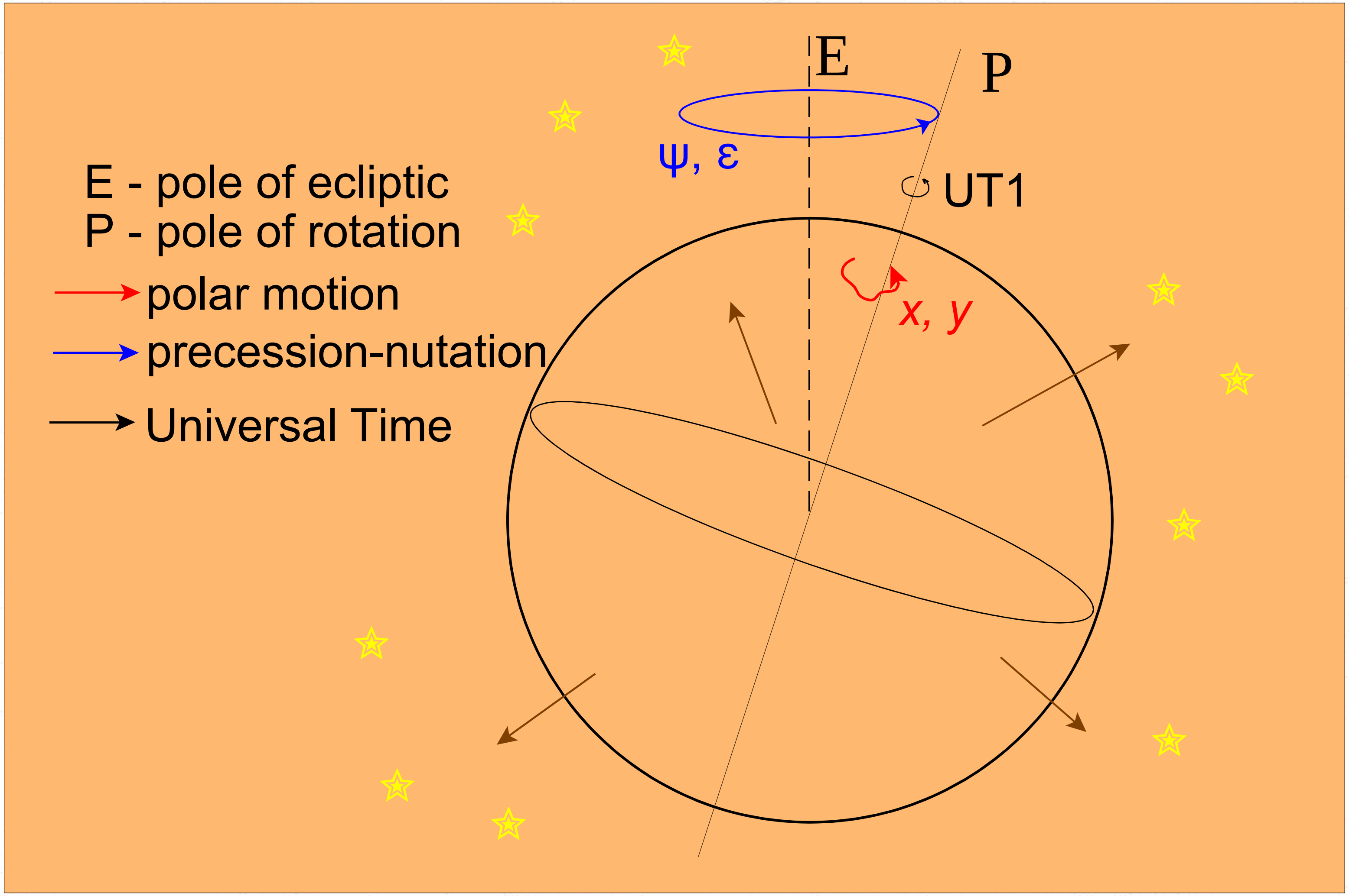
Short history:

- **precession** was known to Hipparchos (2nd cent. BC);
- **polar motion** theoretically predicted by Euler (1765), observationally confirmed by Küstner (1884/5), 2 main components determined by Chandler (1891), since 1899 International Latitude Service (ILS) created;
- **nutation** observed by Bradley and theoretically explained by Euler (middle of 18th cent.);
- **variations in speed of rotation**: effects of secular deceleration in lunar motion observed by Haley (1695), later studied by Laplace (18th cent.), and influence on Earth's rotation implied by G. Darwin (end of 19th cent.). Only in the first half of 20th century decade and seasonal variations were confirmed observationally.

SOLUTION OF THEORY OF EARTH ROTATION:

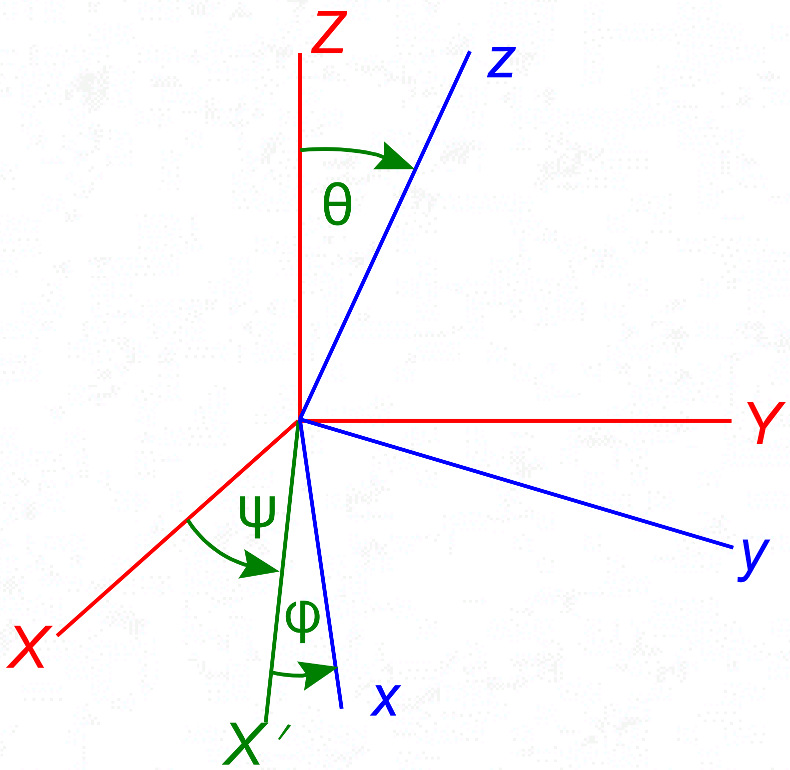
- **rotation of the rigid Earth under the influence of external bodies (Moon, Sun, planets);**
- **influence of modelable non-rigid parts of the Earth (elastic mantle, fluid outer core, solid inner core ...) frequency-dependent transfer function;**
- **not well modelable influences of transfer of mass (atmosphere, oceans, groundwater, changes at core-mantle boundary).**

Earth orientation parameters:



Rotation of the rigid Earth

Changing relation between two reference systems:
xyz - rotating, linked with the Earth
XYZ - non-rotating, linked with extragalactic objects



Euler angles:
- -ψ precession angle
- θ nutation angle
- φ proper rotation angle

Rate of change of the vector of angular momentum = external torque; in a rotating system *xyz*:

$$\frac{d\boldsymbol{H}}{dt} + \boldsymbol{\omega} \times \boldsymbol{H} = \boldsymbol{L}$$

For a general rigid body $\boldsymbol{H} = \boldsymbol{C\omega} + \boldsymbol{h}$, and hence Liouville equations

$$\frac{d}{dt}(\boldsymbol{C\omega} + \boldsymbol{h}) + \boldsymbol{\omega} \times (\boldsymbol{C\omega} + \boldsymbol{h}) = \boldsymbol{L}$$

where $\boldsymbol{C}$ is the tensor of inertia, $\boldsymbol{\omega}$ vector of rotation, $\boldsymbol{h}$ relative angular momentum

For a rigid body holds $\boldsymbol{C}$ = const., $\boldsymbol{h}$ = 0, thus

$$\boldsymbol{C} = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{pmatrix}$$

for rotationally symmetric body, axes oriented to the directions of principal moments of inertia

⇒ Euler dynamical equations (solving rotation in terrestr. syst. *xyz*):

$$\begin{aligned} A\dot{\omega}_1 + & (C - A)\omega_2\omega_3 &= L_1 \\ A\dot{\omega}_2 + & (A - C)\omega_1\omega_3 &= L_2 \\ C\dot{\omega}_3 + & (C - A)\omega_1\omega_2 &= L_3 \end{aligned}$$

ω_1, ω_2 are very small (compared to ω_3), $\omega_3 \approx \Omega$ is almost constant

$$m_1 = \omega_1/\Omega,\ m_2 = \omega_2/\Omega$$
$$m_3 = \omega_3/\Omega - 1 \qquad \approx 10^{-6}$$

Linearized Euler dynamical equations:

$$
\begin{aligned}
m_1 - \dot{m}_2 \frac{A}{\Omega(C-A)} &= -\frac{L_2}{\Omega^2(C-A)} \\
m_2 + \dot{m}_1 \frac{A}{\Omega(C-A)} &= \frac{L_1}{\Omega^2(C-A)} \\
\dot{m}_3 &= \frac{L_3}{\Omega C}
\end{aligned}
$$

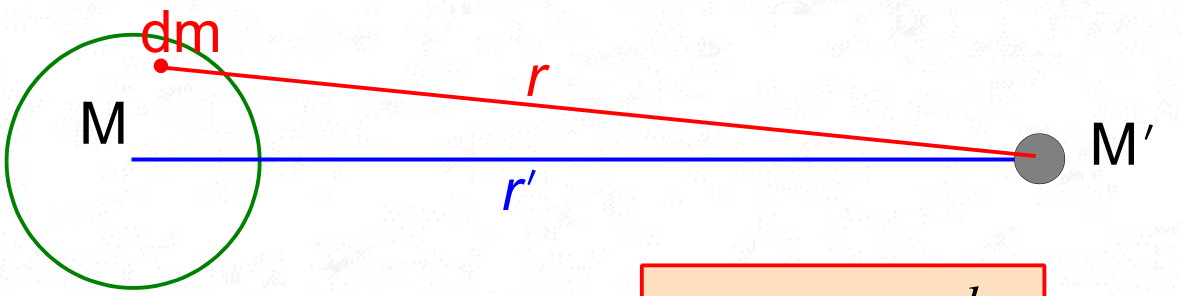
and their homogeneous solution (without external torques):

$$
\begin{aligned}
m_1 &= \gamma \cos(\sigma_E t + \beta) \\
m_2 &= \gamma \sin(\sigma_E t + \beta) \\
m_3 &= konst.
\end{aligned}
$$

where $\sigma_E = \Omega\,(C - A)/A$ is so called Euler period ≈ 305 days,

γ, β are integration constants

External torque $\boldsymbol{L}$ (Moon, Sun, planets):



Force function: $$U = GM' \int_M \frac{dm}{r}$$

$$\boldsymbol{L} = \boldsymbol{r'} \times grad U \quad \begin{cases} L_1 = z'\dfrac{\partial U}{\partial y'} - y'\dfrac{\partial U}{\partial z'} \\ L_2 = x'\dfrac{\partial U}{\partial z'} - z'\dfrac{\partial U}{\partial x'} \\ L_3 = y'\dfrac{\partial U}{\partial x'} - x'\dfrac{\partial U}{\partial y'} \end{cases}$$

Development of external torque into series (if the motions of the Moon, Sun, planets are known):

$$L_1 = (A - C)\sum_{j=1}^{\infty}[A_j^+ \cos(\varphi + \alpha_j t + \beta_j) + A_j^- \cos(\varphi - \alpha_j t - \beta_j)]$$

$$L_2 = (C - A)\sum_{j=1}^{\infty}[A_j^+ \sin(\varphi + \alpha_j t + \beta_j) + A_j^- \sin(\varphi - \alpha_j t - \beta_j)]$$

$$L_3 = 0$$

rapid variable (φ), slow variables ($\alpha_j t$), constants (β_j)

First two Euler equations (polar motion) in complex notation ($\boldsymbol{L} = L_1 + \mathrm{i}L_2$, $\boldsymbol{m} = m_1 + \mathrm{i}m_2$):

$$\boldsymbol{L} = -(C - A)\sum_{j=1}^{\infty}[A_j^+ \mathrm{e}^{-\mathrm{i}(\varphi + \alpha_j t + \beta_j)} + A_j^- \mathrm{e}^{-\mathrm{i}(\varphi - \alpha_j t - \beta_j)}]$$

$$\boldsymbol{m} + \mathrm{i}\dot{\boldsymbol{m}} / \sigma_E = \mathrm{i}\boldsymbol{L} / \Omega^2 (C - A)$$

General solution of Euler equations (motion of the axis of the instantaneous rotation in the body):

$$\boldsymbol{m}(t) = \boldsymbol{m}_\circ \mathrm{e}^{\mathrm{i}\sigma_E t} - \mathrm{i}\frac{\sigma_E}{\Omega^2}\sum_{j=1}^{\infty}\left[\frac{A_j^+}{\Omega + \sigma_E + \alpha_j}\mathrm{e}^{-\mathrm{i}(\varphi + \alpha_j t + \beta_j)} + \frac{A_j^-}{\Omega + \sigma_E - \alpha_j}\mathrm{e}^{-\mathrm{i}(\varphi - \alpha_j t - \beta_j)}\right]$$

Free prograde circular motion with period of 305 days, *ampl.* ≈ 0.4″

Forced retrograde motions with periods of about 1 day *ampl.* ≈ 0.02″, periodically changing

Motion of the axis of angular momentum of the rotating Earth

$\boldsymbol{h}(t) = A\boldsymbol{m}(t)/C \doteq 0.99672\,\boldsymbol{m}(t)$

⇒ axes z (origin), R (inst. rotation) and H (angular momentum) lie always in one plane!

Motion of axes z, R and H in space

Euler kinematical equations (relation between Euler angles and components of vector of rotation):

$$\dot\psi \sin\theta = -\omega_1 \sin\varphi - \omega_2 \cos\varphi = -\Omega\,(m_1 \sin\varphi + m_2 \cos\varphi)$$

$$\dot\theta = -\omega_1 \cos\varphi + \omega_2 \sin\varphi = \Omega\,(-m_1 \cos\varphi + m_2 \sin\varphi)$$

$$\dot\varphi = \omega_3 - \dot\psi \cos\theta = \Omega\,(m_3 + 1) - \dot\psi \cos\theta$$

After substituting m_i and integration we get motion of axis z in space:

$$(\psi - \psi_\circ)\sin\theta = \gamma \frac{A}{C}\cos(\varphi + \sigma_E t + \beta) + \frac{C-A}{\Omega C} A_\circ t + \frac{C-A}{A}\sum_{j=1}^{\infty}\frac{1}{\alpha_j}\left(\frac{A_j^+}{\Omega + \sigma_E + \alpha_j} + \frac{A_j^-}{\Omega + \sigma_E - \alpha_j}\right)\sin(\alpha_j t + \beta_j)$$

$$\theta - \theta_\circ = -\gamma \frac{A}{C}\sin(\varphi + \sigma_E t + \beta) - \frac{C-A}{A}\sum_{j=1}^{\infty}\frac{1}{\alpha_j}\left(\frac{A_j^+}{\Omega + \sigma_E + \alpha_j} - \frac{A_j^-}{\Omega + \sigma_E - \alpha_j}\right)\cos(\alpha_j t + \beta_j)$$

$$\varphi - \varphi_\circ = \Omega t - \psi \cos\theta$$

free motion

precession

nutation

rotation

Motion of axis R in space:

from relations

$$(\psi_R - \psi)\sin\theta = m_2 \sin\varphi - m_1 \cos\varphi$$
$$\theta_R - \theta = m_1 \sin\varphi + m_2 \cos\varphi$$

substituting to preceding equations (for the motion of axis z) we get

$$(\psi_R - \psi_\circ)\sin\theta = \gamma \frac{C-A}{C}\cos(\varphi + \sigma_E t + \beta) + \frac{C-A}{\Omega C} A_\circ t +$$
$$+ \frac{C-A}{A}\sum_{j=1}^{\infty}\frac{1}{\alpha_j}\left[\frac{A_j^+(1+\alpha_j/\Omega)}{\Omega + \sigma_E + \alpha_j} + \frac{A_j^-(1-\alpha_j/\Omega)}{\Omega + \sigma_E - \alpha_j}\right]\sin(\alpha_j t + \beta_j)$$

\- -

$$\theta_H - \theta_\circ = -\gamma \frac{C-A}{C}\sin(\varphi + \sigma_E t + \beta) -$$
$$- \frac{C-A}{A}\sum_{j=1}^{\infty}\frac{1}{\alpha_j}\left[\frac{A_j^+(1+\alpha_j/\Omega)}{\Omega + \sigma_E + \alpha_j} - \frac{A_j^-(1-\alpha_j/\Omega)}{\Omega + \sigma_E - \alpha_j}\right]\cos(\alpha_j t + \beta_j)$$

Motion of axis of angular momentum H in space:

from the equations

$$(\psi_H - \psi)\sin\theta = \frac{A}{C}(m_2 \sin\varphi - m_1 \cos\varphi)$$

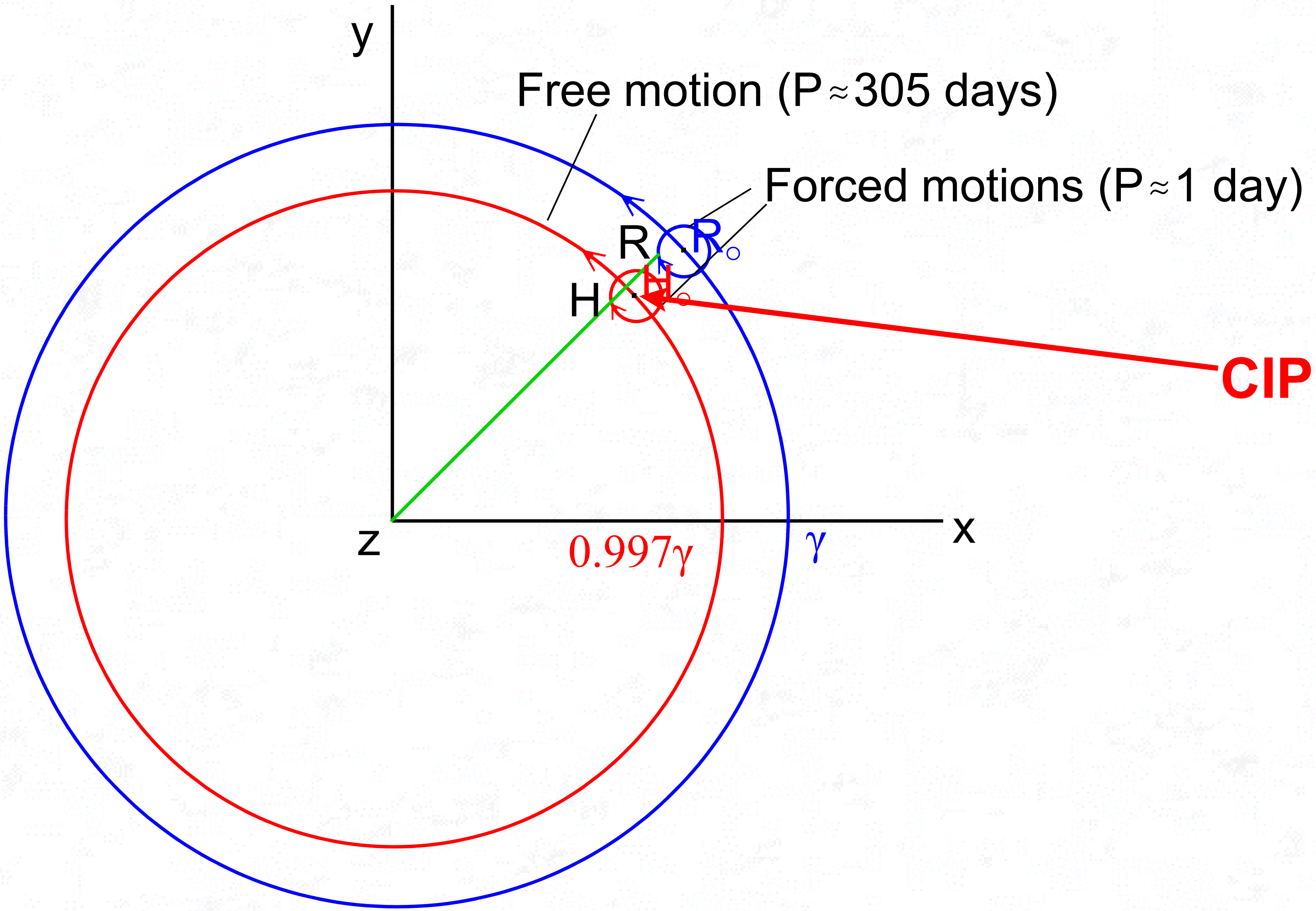
$$\theta_H - \theta = \frac{A}{C}(m_1 \sin\varphi + m_2 \cos\varphi)$$

we get similarly

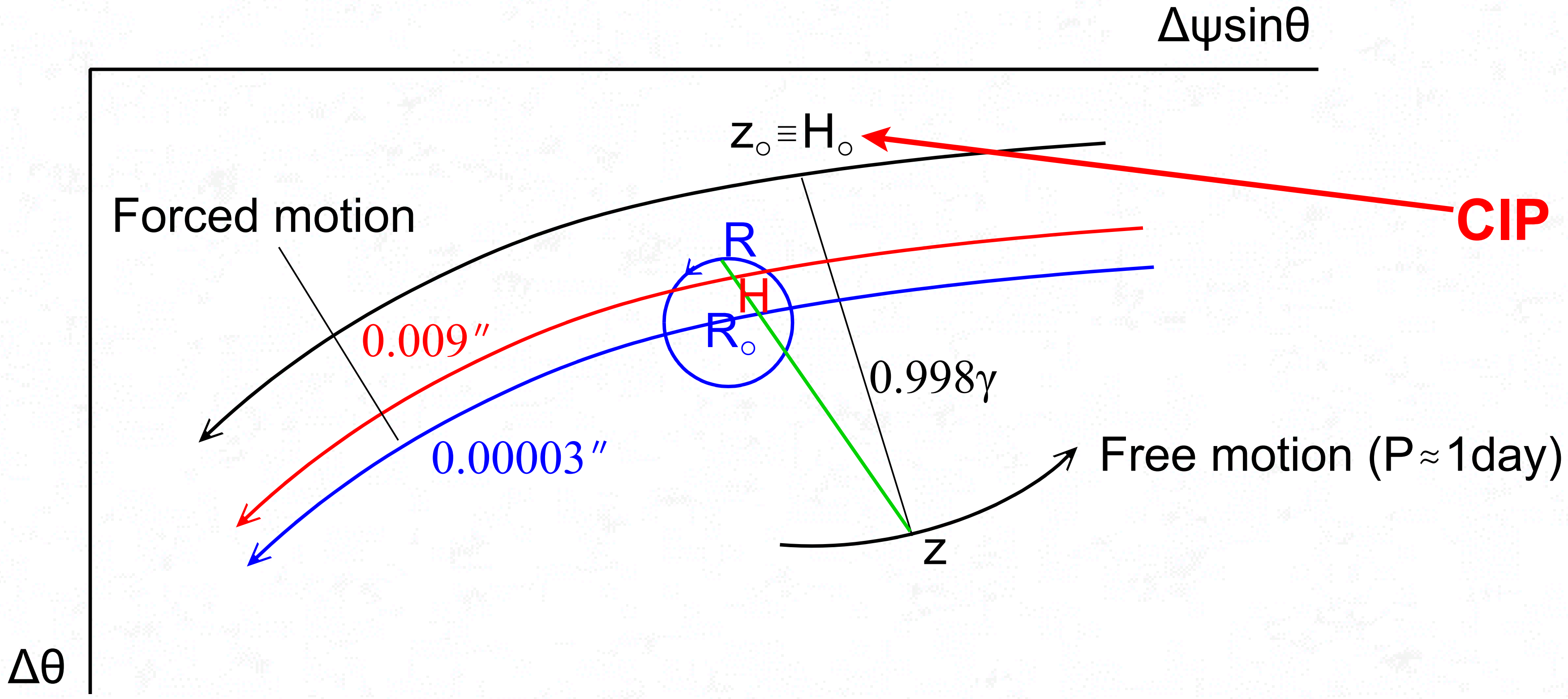
$$(\psi_H - \psi_\circ)\sin\theta = \frac{C-A}{\Omega C} A_\circ t + \frac{C-A}{\Omega C}\sum_{j=1}^{\infty}\frac{1}{\alpha_j}\left(A_j^+ + A_j^-\right)\sin(\alpha_j t + \beta_j)$$

$$\theta_H - \theta_\circ = -\frac{C-A}{\Omega C}\sum_{j=1}^{\infty}\frac{1}{\alpha_j}\left(A_j^+ - A_j^-\right)\cos(\alpha_j t + \beta_j)$$

Motion of axes z, R, H of rigid Earth in the body



Motion of axes z, R, H of rigid Earth in space



Theory of rotation of non-rigid Earth

Liouville equations (for Tisserand axes *xyz*):

$$\frac{d}{dt}(\boldsymbol{C}\omega + \boldsymbol{h}) + \omega \times (\boldsymbol{C}\omega + \boldsymbol{h}) = \boldsymbol{L}$$

where

$$\boldsymbol{C} = \begin{pmatrix} A + c_{11} & c_{12} & c_{13} \\ c_{12} & A + c_{22} & c_{23} \\ c_{13} & c_{23} & C + c_{33} \end{pmatrix} \qquad \boldsymbol{h} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

$\boldsymbol{C}$ is the tensor of inertia, $\boldsymbol{\omega}$ vector of rotation, $\boldsymbol{h}$ relative angular momentum, c_{ij} temporal changes of tensor of inertia

Denoting $h = h_1 + \mathrm{i}h_2, \quad c = c_{13} + \mathrm{i}c_{23}, \quad L = L_1 + \mathrm{i}L_2$

$$\Psi = \left[\Omega^2 c + \Omega h - \mathrm{i}(\Omega \dot{c} + \dot{h} - L)\right] / \Omega^2 (C - A)$$ (excitation functions)

$$\Psi_3 = -(\Omega \dot{c}_{33} + \dot{h}_3 - L_3) / C\Omega$$

we get linearized Liouville equations in **complex form**:

$$\boldsymbol{m} + \mathrm{i}\dot{\boldsymbol{m}} / \sigma_E = \Psi$$

$$\dot{m}_3 = \dot{\Psi}_3$$

Parameters c express the influence of rotational and tidal deformations,

h influence of the wind, oceanic currents etc..

Elastic Earth

Rotational deformations (depending on m_1, m_2, m_3):

$$c_R = \frac{\Omega^2 k a^5}{3G} m$$

$$c_{11R} = c_{22R} = -0.5 c_{33R} = -\frac{2\Omega^2 k a^5}{9G} m_3$$

⇒ changes of position of axis of rotation

⇒ changes of speed of rotation

Tidal deformations (depending on the same arguments as external torques $\boldsymbol{L}$); if it holds:

$$L_1 + \mathrm{i}L_2 = -(C - A)\sum a_j e^{-\mathrm{i}(\varphi + \alpha_j t + \beta_j)}$$

$$L_3 = 0$$

the tidal changes of the tensor of inertia are

$$c_S = \mathrm{i}\frac{ka^5}{3G}\sum a_j e^{-\mathrm{i}(\alpha_j t+\beta_j)}$$

⇒ changes of position of the axis of rotation

$$c_{33S} = -2\frac{ka^5}{3G}\sum b_j \cos(\alpha_j t + \beta_j)$$

⇒ changes of speed of rotation

if we denote

$$k_s = \frac{3G(C-A)}{\Omega^2 a^5} \approx 0.93932$$

'secular' Love number

$$\sigma_C = \sigma_E \frac{k_s - k}{k_s + k\sigma_E/\Omega}$$

Chandler frequency ($P \approx 440$d)

We move rotational deformations to l.h.s. of L. eqs., keep tidal deformations on r.h.s. (in excit. function) we arrive to solution:

$$\boldsymbol{m}(t) = \boldsymbol{m}_\circ \mathrm{e}^{\mathrm{i}\sigma_C t} - \mathrm{i}\frac{\sigma_E}{\Omega^2}\sum_{j=1}^{\infty}\left[\eta(\alpha_j)\frac{A_j^+}{\Omega+\sigma_E+\alpha_j}\mathrm{e}^{-\mathrm{i}(\varphi+\alpha_j t+\beta_j)} + \eta(-\alpha_j)\frac{A_j^-}{\Omega+\sigma_E-\alpha_j}\mathrm{e}^{-\mathrm{i}(\varphi-\alpha_j t-\beta_j)}\right]$$

$$m_3(t) = 2\frac{k(C-A)}{k_s\Omega C}\left[1-\frac{4k(C-A)}{3k_s C}\right]\sum_{j=1}^{\infty} B_j\cos(\alpha_j t+\beta_j)$$

changes with respect to rigid-Earth solution

$$\eta(\alpha) = \frac{(\Omega+\sigma_E+\alpha)(\Omega+k\alpha/k_s)}{(\Omega+\sigma_C+\alpha)(\Omega+k\sigma_E/k_s)} \approx 1+\frac{\alpha}{\Omega}\left(0.30875-\frac{0.00070}{1.0023+\alpha/\Omega}\right)$$

transfer function for elastic Earth

$$\boldsymbol{h}(t) = [A\boldsymbol{m}(t)+\boldsymbol{c}]/C$$
$$\boldsymbol{f}(t) = \boldsymbol{c}/(C\text{-}A)$$

motion of angular momentum axis H

motion of maxim. moment of inertia F $\neq$z

Motion of axes z, R, H and F in space

(similarly as for rigid Earth we use the relation between Euler angles and components of the vector of rotation):

axis z:

$$(\psi - \psi_\circ)\sin\theta = \gamma\frac{\Omega}{\Omega + \sigma_C}\cos(\varphi + \sigma_E t + \beta) + \frac{C-A}{\Omega C}A_\circ t +$$

$$+\frac{C-A}{A}\sum_{j=1}^{\infty}\frac{1}{\alpha_j}\left[\frac{\eta(\alpha_j)A_j^+}{\Omega + \sigma_E + \alpha_j} + \frac{\eta(-\alpha_j)A_j^-}{\Omega + \sigma_E - \alpha_j}\right]\sin(\alpha_j t + \beta_j)$$

$$\theta - \theta_\circ = -\gamma\frac{\Omega}{\Omega + \sigma_C}\sin(\varphi + \sigma_E t + \beta) -$$

$$-\frac{C-A}{A}\sum_{j=1}^{\infty}\frac{1}{\alpha_j}\left[\frac{\eta(\alpha_j)A_j^+}{\Omega + \sigma_E + \alpha_j} - \frac{\eta(-\alpha_j)A_j^-}{\Omega + \sigma_E - \alpha_j}\right]\cos(\alpha_j t + \beta_j)$$

changes with respect to rigid-Earth solution

axis R:

$$(\psi - \psi_\circ)\sin\theta = \gamma \frac{\sigma_C}{\Omega + \sigma_C}\cos(\varphi + \sigma_E t + \beta) + \frac{C-A}{\Omega C} A_\circ t +$$

$$+ \frac{C-A}{A}\sum_{j=1}^{\infty}\frac{1}{\alpha_j}\left[\frac{\eta(\alpha_j)(1+\alpha_j/\Omega)A_j^+}{\Omega + \sigma_E + \alpha_j} + \frac{\eta(-\alpha_j)(1-\alpha_j/\Omega)A_j^-}{\Omega + \sigma_E - \alpha_j}\right]\sin(\alpha_j t + \beta_j)$$

$$\theta - \theta_\circ = -\gamma \frac{\sigma_C}{\Omega + \sigma_C}\sin(\varphi + \sigma_E t + \beta) -$$

$$- \frac{C-A}{A}\sum_{j=1}^{\infty}\frac{1}{\alpha_j}\left[\frac{\eta(\alpha_j)(1+\alpha_j/\Omega)A_j^+}{\Omega + \sigma_E + \alpha_j} - \frac{\eta(-\alpha_j)(1-\alpha_j/\Omega)A_j^-}{\Omega + \sigma_E - \alpha_j}\right]\cos(\alpha_j t + \beta_j)$$

axis H (same as for rigid Earth):

$$(\psi_H - \psi_\circ)\sin\theta = \frac{C-A}{\Omega C} A_\circ t + \frac{C-A}{\Omega C}\sum_{j=1}^{\infty}\frac{1}{\alpha_j}\left(A_j^+ + A_j^-\right)\sin(\alpha_j t + \beta_j)$$

$$\theta_H - \theta_\circ = -\frac{C-A}{\Omega C}\sum_{j=1}^{\infty}\frac{1}{\alpha_j}\left(A_j^+ - A_j^-\right)\cos(\alpha_j t + \beta_j)$$

axis F (substantially different!!):

$$(\psi-\psi_{\circ})\sin\theta=\gamma\left(\frac{\Omega}{\Omega+\sigma_C}-\frac{k}{k_s}\right)\cos(\varphi+\sigma_E t+\beta)+\frac{C-A}{\Omega C}A_{\circ}t+$$

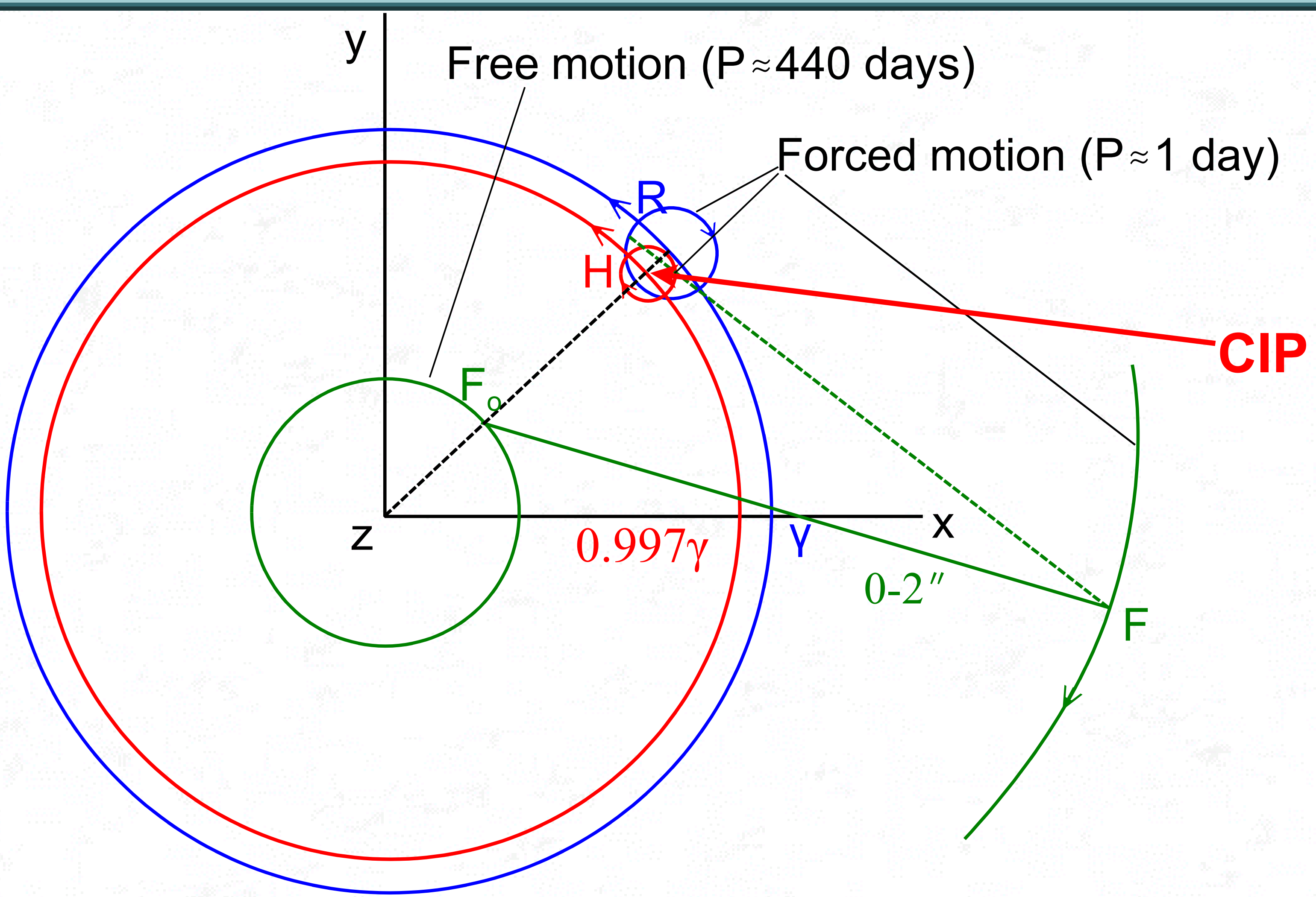
$$+\frac{C-A}{A}\sum_{j=1}^{\infty}\left\{\frac{A_j^+}{\Omega+\sigma_E+\alpha_j}\left[\frac{\eta(\alpha_j)}{\alpha_j}-\frac{kA(1+\alpha_j/\Omega)}{k_s(C-A)\Omega^2}\right]+\frac{A_j^-}{\Omega+\sigma_E-\alpha_j}\left[\frac{\eta(-\alpha_j)}{\alpha_j}+\frac{kA(1-\alpha_j/\Omega)}{k_s(C-A)\Omega^2}\right]\right\}\sin(\alpha_j t+\beta_j)$$

- -

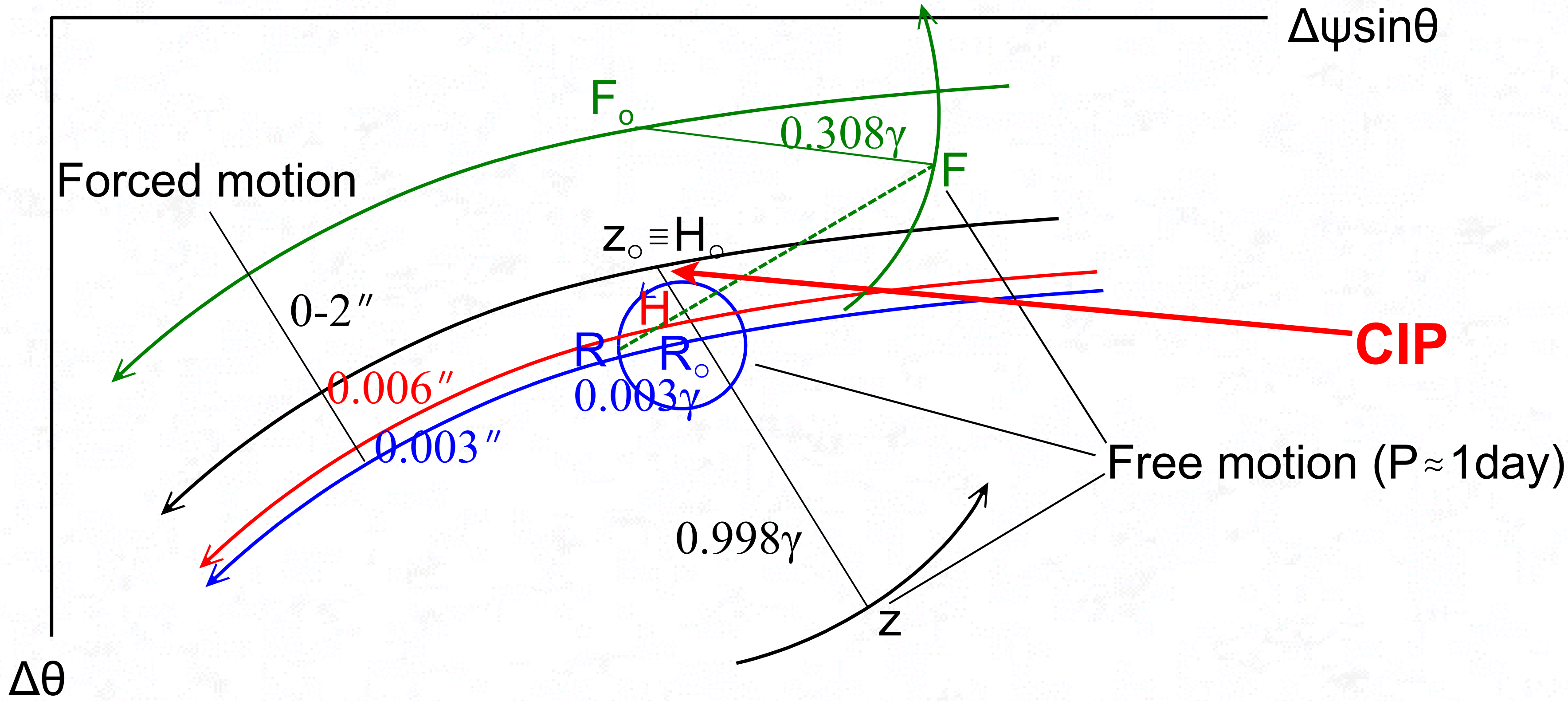
$$\theta-\theta_{\circ}=-\gamma\left(\frac{\Omega}{\Omega+\sigma_C}-\frac{k}{k_s}\right)\sin(\varphi+\sigma_E t+\beta)-$$

$$-\frac{C-A}{A}\sum_{j=1}^{\infty}\left\{\frac{A_j^+}{\Omega+\sigma_E+\alpha_j}\left[\frac{\eta(\alpha_j)}{\alpha_j}-\frac{kA(1+\alpha_j/\Omega)}{k_s(C-A)\Omega^2}\right]-\frac{A_j^-}{\Omega+\sigma_E-\alpha_j}\left[\frac{\eta(-\alpha_j)}{\alpha_j}+\frac{kA(1-\alpha_j/\Omega)}{k_s(C-A)\Omega^2}\right]\right\}\cos(\alpha_j t+\beta_j)$$

Motion of axes R, H, F of elastic Earth in the body



Motion of axes z, R, H, F of elastic Earth in space



Influence of fluid flattened core

Poincaré model (rigid mantle + fluid core):

$$
\begin{aligned}
A\dot{\omega}_1 + F\dot{\chi}_1 &+ (C-A)\Omega\,\omega_2 - F\Omega\,\chi_2 &= L_1 \\
A\dot{\omega}_2 + F\dot{\chi}_2 &- (C-A)\Omega\,\omega_1 + F\Omega\,\chi_1 &= L_2 \\
F\dot{\omega}_1 + A_c\dot{\chi}_1 &- C_c\Omega\,\chi_2 &= 0 \\
F\dot{\omega}_2 + A_c\dot{\chi}_2 &+ C_c\Omega\,\chi_1 &= 0
\end{aligned}
$$

A, C princ. mom. inert. of the whole body

A_c, C_c - " - Core ($C_c \approx 0.115C$)

$F = 2M_c ac \approx A_c$

χ_1, χ_2 components of rotation of the core wrt mantle

Homogeneous solution (L=0) yields two free frequencies:

$$\sigma_1 = \Omega \frac{C-A}{A-A_c} \approx 0.0233 \text{ rad / day}$$

$$\sigma_2 = -\Omega\left[1 + \varepsilon C / (C - C_c)\right] \approx -6.3187 \text{ rad / day},$$

$$\text{where } \varepsilon = \frac{C_c - A_c}{C_c} \approx 0.0026$$

$P \approx$ 270 days - 'Chandler' per.

$P \approx$ 23h 52min - FCN, NDFW

Dynamical flattening of the core

Solution ($\boldsymbol{m}_1$, $\boldsymbol{m}_2$ are integration constants):

$$\boldsymbol{m}(t) = \boldsymbol{m}_1 \mathrm{e}^{-\mathrm{i}(\sigma_1 t+\beta_1)} + \boldsymbol{m}_2 \mathrm{e}^{-\mathrm{i}(\sigma_2 t+\beta_2)} + \text{forced terms}$$

$$\chi_1 + \mathrm{i}\chi_2 = \boldsymbol{m}_1 \frac{A_c\sigma_1}{A_c\sigma_1 + \Omega C_c} \mathrm{e}^{-\mathrm{i}(\sigma_1 t+\beta_1)} - \boldsymbol{m}_2 \frac{A_c\sigma_2}{A_c\sigma_2 + \Omega C_c} \mathrm{e}^{-\mathrm{i}(\sigma_2 t+\beta_2)} + \text{forced terms}$$

Forced terms have the same form as for the rigid Earth, but the transfer function is:

$$\eta(\alpha) = \left[1 - \frac{A_c\alpha(\Omega+\alpha)}{A(\Omega+\sigma_E+\alpha)(\alpha-\varepsilon\Omega)}\right]^{-1} \approx$$

$$\approx \left[1 - \frac{0.112(1+\alpha/\Omega)}{(1.003285+\alpha/\Omega)(1-0.0026\Omega/\alpha)}\right]^{-1}$$

Combined influence of fluid core and elastic mantle

$$\begin{aligned}
A\dot{\omega}_1 + (C-A)\Omega\,\omega_2 \quad &+ A_c(\dot{\chi}_1 - \Omega\,\chi_2) + \Omega\,(\dot{c}_{13} - \Omega\,c_{23}) &= L_1\\
A\dot{\omega}_2 - (C-A)\Omega\,\omega_1 \quad &+ A_c(\dot{\chi}_2 + \Omega\,\chi_1) + \Omega\,(\dot{c}_{33} - \Omega\,c_{13}) &= L_2\\
A_c\dot{\omega}_1 + A_c\dot{\chi}_1 \quad &- C_c\Omega\,\chi_2 + \Omega\,\dot{c}_{c13} &= 0\\
A_c\dot{\omega}_2 + A_c\dot{\chi}_2 \quad &+ C_c\Omega\,\chi_1 + \Omega\,\dot{c}_{c23} &= 0
\end{aligned}$$

Solution formally identical with preceding models, different is the definition of transfer function η:

$$\eta(\alpha) = 1 - \frac{\alpha}{\Omega}\left(1.00328 + \frac{\alpha}{\Omega}\right) \times$$
$$\times\left[0.416 + \left(0.073 + \frac{\alpha}{\Omega}\right)\left(\frac{1.06}{1.00328 + \alpha/\Omega} - \frac{0.810}{1.00248 + \alpha/\Omega} + \frac{0.665}{0.0021714 - \alpha/\Omega}\right)\right]$$

Wahr, IAU1980

- **Additional consequences of Wahr model:**
  - **Free polar motion (Chandler period):**
    - ***P*** **= 400 days (in terrestrial system);**
  - **Free Core Nutation (FCN):**
    - ***P*** **= 460 days (in celestial system).**

Adding inner solid core, viscosity of the mantle...

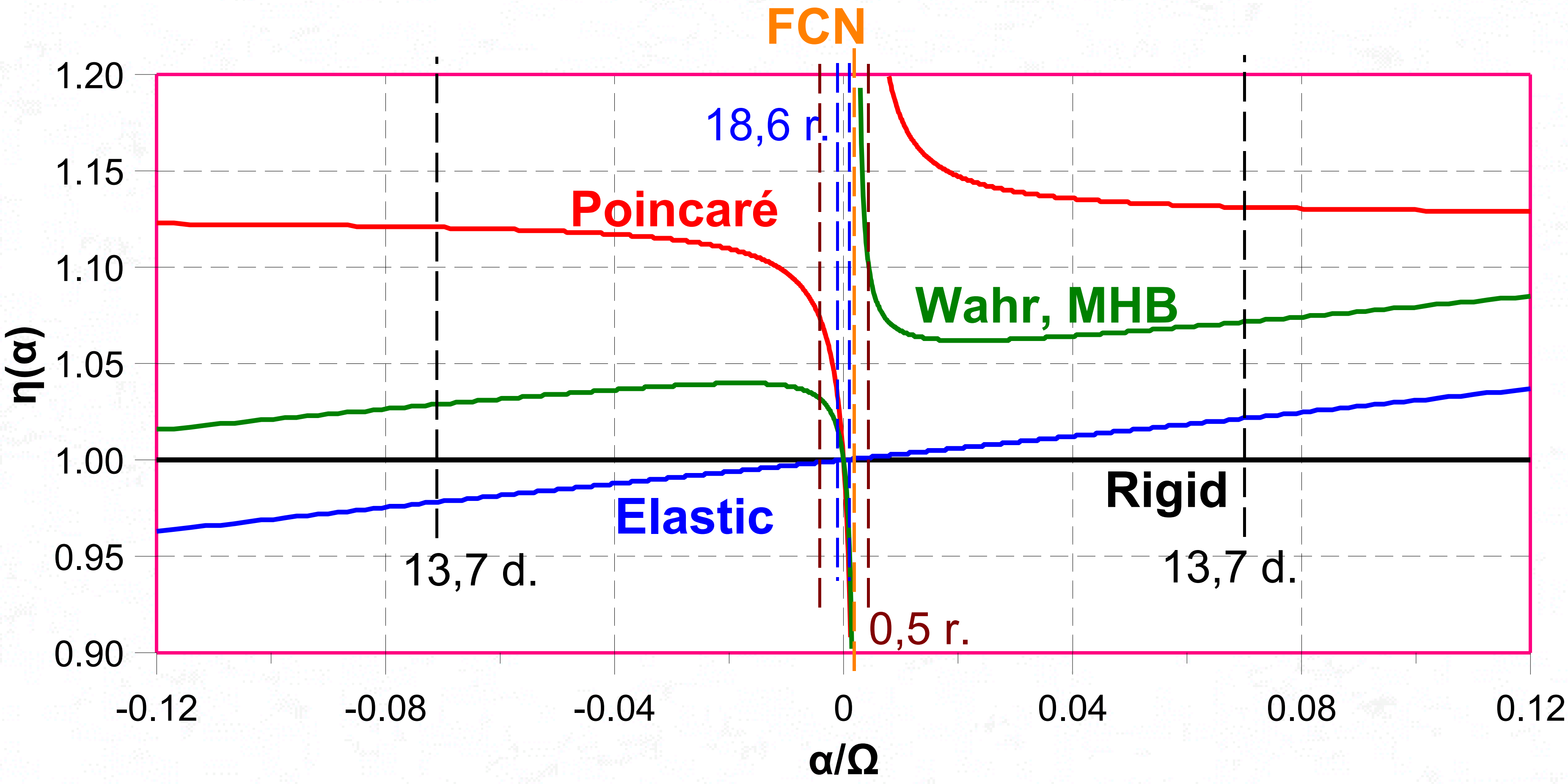
MHB (complex) transfer function:

$$\eta(\sigma) = \frac{e_R - \sigma}{e_R + 1} N_0 \left[1 + (1+\sigma)\left(Q_0 + \sum_{j=1}^{4} \frac{Q_j}{\sigma - s_j} \right) \right]$$

where e_R is dynamical ellipticity of rigid Earth, $\sigma = -1 - \alpha/\Omega$ is the frequency of nutation (in ITRF), N, Q are constants, and s_j are resonance frequencies:

1. Chandler Wobble - CW ($P_{ter.}$ = 435 d);
2. Retrograde Free Core Nutation - RFCN ($P_{cel.}$ = 430 d);
3. Prograde Free Core Nutation - PFCN ($P_{cel.}$ = 1020 d);
4. Inner Core Wobble - ICW ($P_{ter.}$ = 2400 d).

Transfer function for different models



Additional influences (geophysical):

- **Influence of the oceans:**
  - Longer period of Chandler wobble by app. 35 days;
  - Excitation of Chandler wobble;
  - Forced polar motion (seasonal, diurnal);
  - Excitation of FCN ...
- **Influence of the atmosphere:**
  - Forced polar motion (pressure changes);
  - Forced changes of speed of rotation (zonal winds)
  - Excitation of FCN ...
- **Viscosity of the mantle:**
  - Damping of Chandler wobble;
  - Phase change of nutation terms ...

Recapitulation:

- **External forces (solar system bodies):**
  - ♦ **Dominant influence on precession and nutation;**
  - ♦ **Partial influence on speed of rotation (tidal deformations);**
- **Internal forces (geophysical):**
  - ♦ **Dominant influence on polar motion (changes of tensor of inertia);**
  - ♦ **Partial influence on speed of rotation (changes of tensor of inertia);**
- **Precession depends only marginally on the adopted model of internal structure of the Earth (only on global moments of inertia).**

Models of nutation used in astronomy:

- **Until 1980 Woolard (1953):**
  - Rigid Earth;
  - Influence of only Moon and Sun;
  - 69 terms.
- **1984-2002 IAU1980 (Kinoshita / Wahr 1977, 1979):**
  - Non-rigid Earth (stratified elastic mantle, fluid core);
  - Influence of only Moon and Sun;
  - 106 terms.
- **2003 - IAU2000 (Souchay et al. / Mathews et al.1997, 2002):**
  - Non-rigid Earth (visco-elastic mantle, fluid outer core, solid inner core, atmosphere, ocean, elmg. torques outer core-mantle, inner-outer core);
  - Influence of the Moon, Sun and planets;
  - 1360 terms.