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ON THE GLOBAL REGULARIZING TRANSFORMATIONS OF THE RESTRICTED THREE-BODY PROBLEM

B. ÉRDI

Department of Astronomy, Eötvös University
Budapest, Hungary

Abstract. The uniqueness of the general form $q = f(w) = \left[h(w) + h(w)^{-1}\right]/4$ of the global regularizing transformations of the restricted three-body problem is shown.

1. INTRODUCTION

In the restricted problem of three bodies two bodies revolve around their center of mass in circular orbits under the influence of their mutual gravitational attraction and a third body, influenced by only the gravitational attraction of the other two bodies, moves in the plane defined by the revolving bodies. It is assumed that the third body has so small mass that it does not influence the motion of the others. All the three bodies are considered as point masses. The problem is to find the motion of the third body.

The equations of motion of the third body, in properly chosen units, in a barycentric coordinate system which rotates uniformly together with the revolving bodies, are (Szebehely, 1967) :

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x, \\ \ddot{y} + 2\dot{x} &= \Omega_y, \end{aligned} \tag{1}$$

where

$$\Omega = \frac{1}{2}\left[(1-\mu)r_1^2 + \mu r_2^2\right] + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}, \tag{2}$$

$$r_1 = \sqrt{(x-\mu)^2 + y^2}, \quad r_2 = \sqrt{(x+1-\mu)^2 + y^2}.$$

In Equations (1) x, y are the rectangular coordinates of the third body P_3 (see Figure 1), the dots mean differentiation with respect to the time variable t (which is actually the mean anomaly of the revolving bodies, the primaries), x and y in the index mean partial differentiation, μ is the mass parameter (the mass of the smaller primary divided by the total mass of the primaries), the body P_1 has mass $1-\mu$ and coordinates $(\mu, 0)$, the body P_2 has mass μ and coordinates $(\mu - 1, 0)$, r_1 and r_2 are the distances of P_3 from P_1 and P_2.

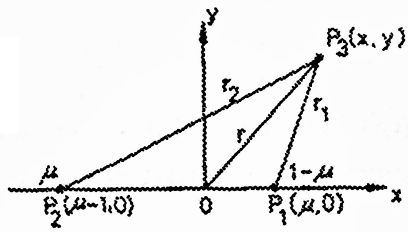

Fig. 1. **Notations of the problem.**

Equations (1) possess the well known Jacobian integral

$$\dot{x}^2 + \dot{y}^2 = 2\Omega - C, \tag{3}$$

where C is the Jacobian constant.

Equations (1) become singular at collisions of the third body with the primaries when $r_1 = 0$ or $r_2 = 0$. However, these singularities are not of essential character and can be removed by regularizing transformations. An extended summary on the problems and methods of regularization can be found in Hagihara (1975) and in Szebehely (1967).

The regularizing transformations are conveniently written in complex variables. Introducing $z = x + iy$ ($i = \sqrt{-1}$), Equations (1) can be written as

$$\ddot{z} + 2i\dot{z} = \text{grad}_z \Omega, \tag{4}$$

where

$$\text{grad}_z \Omega = \frac{\partial \Omega}{\partial x} + i\frac{\partial \Omega}{\partial y}.$$

By making use of the transformations

$$z = f(w), \tag{5}$$

$$\frac{dt}{d\tau} = \left|\frac{df}{dw}\right|^2, \tag{6}$$

where $w = u + iv$, Equation (4) can be transformed into (Szebehely, 1967)

$$w'' + 2i\left|\frac{df}{dw}\right|^2 w' = \text{grad}_w \Omega^*. \tag{7}$$

Here the prime means differentiation with respect to the new time variable τ and

$$\Omega^* = \left|\frac{df}{dw}\right|^2 \left(\Omega - \frac{C}{2}\right), \tag{8}$$

where in Ω

$$r_1 = |z - \mu|, \quad r_2 = |z + 1 - \mu|.$$

Properly selecting the function $f(w)$, the critical terms $1/r_1$, $1/r_2$ of the potential Ω can be made regular and thus Equation (7) also will be regular.

The Jacobian integral for Equation (7) is

$$|w'|^2 = 2\Omega^*. \tag{9}$$

2. REGULARIZING TRANSFORMATIONS

Several regularizing transformations are known in the literature. The parabolic transformation

$$z = w^2, \tag{10}$$

introduced by Levi-Civita (1904), can regularize a singularity at the origin of the coordinate system. Thus, with a translation of the origin of the coordinate system to either one of the primaries, the collision singularities can be regularized. Accordingly,

$$z = \mu + w^2 \tag{11}$$

will regularize the singularity at P_1 and

$$z = \mu - 1 + w^2 \tag{12}$$

will do that for the singularity at P_2. These transformations are called local regularizations, since they can eliminate only one of the singularities.

There exist transformations which can remove the two singularities together. These are called global regularizing transformations (although in a mathematical sense they are local operations). It is convenient to write these transformations in a coordinate system in which the two primaries are located symmetrically with respect to the origin. The translation

$$q = z + \frac{1}{2} - \mu \tag{13}$$

shifts the origin into the midpoint of the primaries and they will be located at $q = \pm\frac{1}{2}$. Instead of Equation (4) we have

$$\ddot{q} + 2i\dot{q} = \mathrm{grad}_q \Omega \tag{14}$$

with Ω given by Equation (2), but

$$r_1 = \left|q - \frac{1}{2}\right|, \quad r_2 = \left|q + \frac{1}{2}\right|. \tag{15}$$

The transformations

$$q = f(w), \tag{16}$$

$$\frac{dt}{d\tau} = \left|\frac{df}{dw}\right|^2 \tag{6}$$

transform Equation (14) into Equation (7).

The Thiele-Burrau transformation, introduced by Thiele (1895) for $\mu = 0.5$ and generalized by Burrau (1906) for arbitrary μ, can be given as

$$q = \frac{1}{2}\cos w. \tag{17}$$

It is easy to see that in this case $|df/dw|^2 = r_1 r_2$, and thus in Equation (8) the two critical terms are regularized together.

Birkhoff's (1915) transformation can be expressed as

$$q = \frac{1}{4}\left(2w + \frac{1}{2w}\right). \tag{18}$$

In this case $|df/dw|^2 = r_1 r_2 / |w|^2$. This expression eliminates the singularities at P_1 and P_2, but introduces a new singularity, $w = 0$, in the transformed plane. However, $w = 0$ corresponds to $q = \infty$ and thus all points of the finite physical plane are regularized by this transformation.

Birkhoff's transformation can be generalized to (Deprit and Broucke, 1963; Arenstorf, 1963)

$$q = \frac{1}{4}\left(w^2 + \frac{1}{w^2}\right). \tag{19}$$

This transformation is called Lemaître's transformation due to its relation to the regularization of the general problem of three bodies by Lemaître (1955). In this case $|df/dw|^2 = 4r_1 r_2 / |w|^2$.

There exist other generalizations as well. Wintner (1930) generalized Birkhoff's transformation and in the 'midpoint' coordinate system this can be expressed as

$$q = \frac{1}{2}\frac{(w+\frac{1}{2})^{2n} + (w-\frac{1}{2})^{2n}}{(w+\frac{1}{2})^{2n} - (w-\frac{1}{2})^{2n}}, \tag{20}$$

where n is any positive integer.

Broucke (1965) generalized the Thiele-Burrau transformation to

$$q = \frac{1}{2}\cos nw, \tag{21}$$

and Birkhoff's regularization to

$$q = \frac{1}{4}\left(w^n + \frac{1}{w^n}\right), \tag{22}$$

where n is any nonzero real number.

The global regularizing transformations (17)-(22) can be expressed in the general form

$$q = f(w) = \frac{1}{4}\left[h(w) + \frac{1}{h(w)}\right], \tag{23}$$

where $h(w) = e^{iw}$, $2w$, w^2, and e^{inw}, w^n for the Thiele-Burrau, the Birkhoff, the Lemaître and the two Broucke transformations, respectively, while in the case of Wintner's transformation

$$h = \frac{(w+\frac{1}{2})^n + (w-\frac{1}{2})^n}{(w+\frac{1}{2})^n - (w-\frac{1}{2})^n}.$$

One may ask, why Equation (23) is the general form for the global regularization. Is not there any other form which together with the time transformation (6) results in regularization?

3. UNIQUENESS OF THE GENERAL FORM OF GLOBAL REGULARIZATION

To answer the above question we note that in order to eliminate the singularities at $r_1 = 0$ and $r_2 = 0$ we must have

$$\left|\frac{df}{dw}\right|^2 = \gamma(w)(r_1 r_2)^n. \tag{24}$$

Then forming the product $|df/dw|^2\Omega$, the terms $1/r_1$ and $1/r_2$ become regular. In Equation (24) the function $\gamma(w)$ must be regular at the singularities. As to the exponent n, it is necessary that $n = 1$, since $n > 1$ results in $\Omega^* = 0$ at the singularities, and from Equations (9) and (7) it follows that $w' = 0$, $w'' = 0$ at the singularities. That is, a steady-state solution is obtained in this case.

Thus the determining equation for the function $f(w)$ from Equations (24), (15), (16) is

$$\left|\frac{df}{dw}\right|^2 = \gamma(w)\left|f - \frac{1}{2}\right|\left|f + \frac{1}{2}\right|. \tag{25}$$

With $\gamma = |\alpha|^2$, this equation leads to

$$\frac{df}{dw} = \alpha\sqrt{f^2 - \frac{1}{4}}.$$

By integration we obtain

$$\ln(2f + \sqrt{4f^2 - 1}) = \beta,$$

where $\beta = \int \alpha\, dw$ and the arbitrary constant may be set zero. Solving for f, we obtain

$$f = \frac{1}{4}\left(e^\beta + \frac{1}{e^\beta}\right) \tag{26}$$

and with $\beta = \ln h(w)$,

$$f = \frac{1}{4}\left[h(w) + \frac{1}{h(w)}\right]. \tag{23}$$

Thus, Equation (23) represents the unique form for the global regularizing transformations when the time transformation is given by Equation (6).

It also follows that

$$\gamma = \left|\frac{d\beta}{dw}\right|^2 = \left|\frac{\frac{dh}{dw}}{h}\right|^2. \tag{27}$$

The simplest form of Equation (26) is obtained for $\beta = w$ or $\beta = iw$, in both cases $\gamma = 1$. Actually, $\beta = iw$ gives the Thiele-Burrau transformation, $f = \frac{1}{2}\cos w$, while $\beta = w$ results in $f = \frac{1}{2}\cosh w$. Both transformations, written explicitly in the coordinates, offer similar formalism using elliptic coordinates. A similar kind of transformation can be obtained with $h = e^{iw}/i$, which results in $f = \frac{1}{2}\sin w$, $\gamma = 1$. All these transformations can be generalized to $f = \frac{1}{2}\cos nw$ (Broucke, 1965), $f = \frac{1}{2}\cosh nw$, $f = \frac{1}{2}\sin nw$, $\gamma = n^2$, where n is any nonzero real number.

Considering Equation (23), it is natural to look for the simplest form of the function $h(w)$ and thus one may take a linear function $h(w) = aw + b$, with a, b arbitrary constants. Among these transformations the only one which leaves the primaries at their places is obtained for $a = 2$, $b = 0$, and this is Birkhoff's transformations.

Simple as it is but it seems so that Equation (25) is not written in the literature. The use of the type of Equation (25) appears in more general problems. For example, Giacaglia (1967) in order to regularize a four-body problem in which three primaries of arbitrary masses revolve in circular orbits around their center of mass in the equilateral equilibrium configuration and a body of infinitesimal mass moves in their field, studies the equation (changing somewhat his notations)

$$\left|\frac{df}{dw}\right|^2 = |g(w)|^2|f - f_1||f - f_2||f - f_3|.$$

Mavraganis (1988) for the global regularization of the magnetic-binary problem solves the equation

$$\left|\frac{df}{dw}\right|^2 = r_1^3 r_2^3.|hbox$$

For other examples in the more general case see Hagihara (1975).

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