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ON THE ONE TYPE OF THE RESTRICTED THREE BODY PROBLEM

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Summary: One type of the restrected three-body problem is formulated. The first integrals and the exact solutions are considered as they stand to the corrensponding ones in the general three-body problem. Equations of motion in Jacobi coordinates are expanded over Legendre polinomials, which gives suitable form for studing of the so called stelar configurations of the three-body problem.

1. INTRODUCTION

In general three-body problem (GTBP) of classical celestial mechanics subject is the motion of three material points which simultaniously attract each other by Newtonian gravity. It has been proved (Sundman, 1913) that solution of the problem can be expressed in the form of convergent series, but convergence is so slow that solution is practically unusable (Duboshin, 1964). Celestial mechanicists have formulated a few simplier problems (based on GTBP) which have some practical importance and are known as restricted three-body problems (RTBP) (for review see Szebehely 1962).

If mass of one of the three bodies, let's say m_0, is significantly greater than the sum of the other two masses ($m_0 \gg m_1+m_2$, and m_1 and m_2 are of the same order of quantity) than it is reasonable to expect that gravitational influence of bodies m_1 and m_2 on body m_0 will be negligable. In general, gravitational attraction between bodies m_1 and m_2 could be negligable also, and as it was stated by Hénon and Petit (Hénon and Petit, 1986.) than problem reduces on two, practically independent, two - body problems for pairs (m_0-m_1) and (m_0-m_2). However, these authors have also noticed that if relative distance between bodies m_1 and m_2 is "sufficiently small, their mutual attraction becomes of the same order as the differential attraction from m_0", then their mutual attracion can not be neglected, and they classified such types of problems as Hill's type problems.

2. FORMULATION OF THE PROBLEM

In order to simplify GTBP, no matter how distant are bodies m_1 and m_2 from each other, one can entirely neglect the influence of bodies m_1 and m_2 on body m_0 (or fix body m_0 with inertial reference frame). *More generally, no matter how masses m_0, m_1, m_2 are related, one can fix body m_0 in order to obtain a type of RTBP. In that case problem would be to find the motion* *of bodies m_1 and m_2 if they are attracted by fixed body and by each other, and if initial conditions are known.* Comparing with problem of two fixed centers (e.g. Duboshin, 1964), this problem is in some way opposite and could be called problem of one fixed center. On the other side comparing this problem with various versions of RTBP one can see that, it is more complicated since motion of two bodies is not known. Here, we will just make note that type of problems in which one has mass configuration $m_0 \gg m_1 + m_2$ are very often classified as stelar types three body problems (e.g. Roy, 1982).

3. THE EQUATIONS OF MOTION AND THEIR EXACT SOLUTIONS

The equations of motion (notations are taken form fig. 1) in case explained in the previous chapter are:

$$\frac{d^2\mathbf{r}_1}{dt^2} = -Gm_0\frac{\mathbf{r}_1}{{r_1}^3} + Gm_2\frac{\mathbf{r}}{r^3}, \tag{1}$$

$$\frac{d^2\mathbf{r}_2}{dt^2} = -Gm_0\frac{\mathbf{r}_2}{{r_2}^3} - Gm_1\frac{\mathbf{r}}{r^3}, \tag{2}$$

where gravitational accelerations which bodies m_1 and m_2 are giving to the body m_0 are entirely neglected (or say the third equation from GTBP is neglected) because we put $\ddot{\mathbf{r}}_0 = \dot{\mathbf{r}}_0 = \mathbf{r}_0 = 0$. The equations of motion (1) and (2) of bodies (m_1, m_2) posses two first integrals (integral of angular momentum and integral of energy) which could be easily checked by direct calculation.

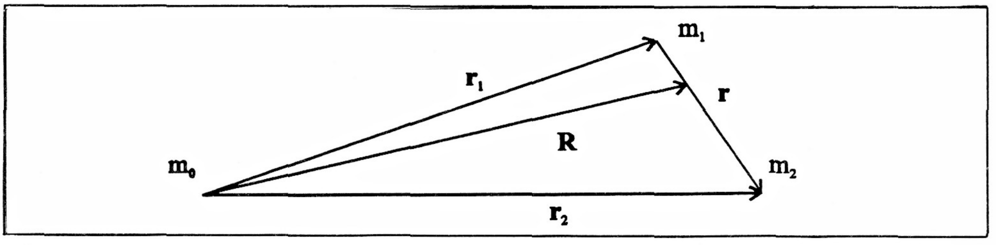

Figure 1.

The conservation laws are valid in this problem as in GTBP, but the integrals of motion of barycenter dissapeared because m_0 is fixed. On the other hand situation with exact solutions is different. While in GTBP two types of the exact solutions exist (colinear and triangle type), in this case one has colinear type only.

It is well known that if center of mass of the bodies m_1 and m_2 coincides with the position of the body m_0, and if initial velocities of bodies m_1 and m_2 are in directions making a fixed and the same angle with their radius vectors respect to the barycenter, than the equations (1) and (2) can be integrated in the finite form. Actually these equations are becoming one and the same equation:

$$\frac{d^2\mathbf{r}_i}{dt^2} = -G\frac{m' + 8m_0}{r_i{}^3}\mathbf{r}_i \quad (i = 1, 2), \tag{3}$$

where $m' = m_1 = m_2$ (that is the consequence of the condition, about coincidence of the body m_0 with the center of mass of the system m_1-m_2, mentioned above). In the equation (3) one can easily recognizes the equation of two body problem. Finally one can conclude that simplified colinear exact solution exists in this type of RTBP. It is simplified because m_0 has to be exactly on the half of the distance between m_1 and m_2 (which is also cosequence of the condition about coincidence of the body m_0 with the center of mass of the system m_1-m_2).

4. THE EQUATIONS OF MOTION IN JACOBI COORDINATES. COMPARISON WITH GTBP

Let us skip now to Hill's configuration of this problem. Relative distance among small bodies (denoted by $\mathbf{r}$) is very small (by magnitude) comparing with distance from their center of mass to the large mass (denoted by $\mathbf{R}$), (see fig. 1). Than we have desirable mass configuration and also $r \ll R$. In such kind of problems when hierarchy in configuration is discrenable, Jacobi coordinates $\mathbf{R}$, $\mathbf{r}$, are very convinient. If we put $m = m_1 + m_2$ and $\mu = m_1 m_2/(m_1 + m_2)$ then equations of motion (1) and (2) in Jacobi coordinates will take the following form:

$$\frac{d^2\mathbf{R}}{dt^2} = -G\frac{m_0}{m}\left[\left(\frac{m_2}{r_2{}^3} + \frac{m_1}{r_1{}^3}\right)\mathbf{R} + \mu\left(\frac{1}{r_2{}^3} - \frac{1}{r_1{}^3}\right)\mathbf{r}\right], \tag{4}$$

$$\frac{d^2\mathbf{r}}{dt^2} + G\frac{m}{r^3}\mathbf{r} = -Gm_0\left[\left(\frac{1}{r_2{}^3} - \frac{1}{r_1{}^3}\right)\mathbf{R} + \frac{1}{m}\left(\frac{m_1}{r_2{}^3} + \frac{m_2}{r_1{}^3}\right)\mathbf{r}\right], \tag{5}$$

If one compares these equations with corrensponding equations of GTBP, in Jacobi coordinates (see Roy, 1982, p.413) only small difference in equation (4) will be noticed (instead of m_0 on the right hand side one has $m_0 + m$, while equation (5) keeps the same form as in GTBP. Another, essential difference, is that noninertial motion of m_0 is neglected, while in GTBP m_0 has noninertial component of motion respect to the barycenter.

Functions in small brackets appearing in equations (4) and (5) can be expanded in series over derivatives of the Legendre's polinomials. If $P_k{}^{(1)}(x)$ is being derivative of Legendres polinomial $P_k(x)$ over x ($x = \cos\varphi$ where φ is angle between vectors $\mathbf{r}$ and $\mathbf{R}$), equations (4) and (5) could be written in the following form:

$$\frac{d^2\mathbf{R}}{dt^2} + G\frac{m_0}{R^3}\mathbf{R} = -G\frac{m_0}{R^3}\frac{\mu}{m}\sum_{k=2}^{\infty}(B_k\mathbf{R} + A_k\mathbf{r})\Delta^{k-1}P_k{}^{(1)}(\cos\varphi), \tag{6}$$

$$\frac{d^2\mathbf{r}}{dt^2} + G\frac{m}{r^3}\mathbf{r} = -G\frac{m_0}{R^3}\mathbf{r} - G\frac{m_0}{R^3}\sum_{k=2}^{\infty}(A_k\mathbf{R} + C_k\mathbf{r})\Delta^{k-1}P_k{}^{(1)}(\cos\varphi), \tag{7}$$

where

$$A_k = \frac{(-1)^{k-1} {m_1}^{k-1} - {m_2}^{k-1}}{m^{k-1}}, \quad B_k = \frac{(-1)^{k-1} {m_1}^{k-2} + {m_2}^{k-2}}{m^{k-2}},$$

$$C_k = \frac{(-1)^{k-1} {m_1}^{k} + {m_2}^{k}}{m^{k}}, \quad (k = 2, 3, ...); \qquad \Delta = \frac{r}{R}.$$

Coeffitients A_k, B_k and C_k are all in the interval $[-1, 1]$ for $k \geq 2$. Condition $\Delta \ll 1$ should provide that bodies m_1 and m_2 will be in binary system which will itself move around m_0.

Difference between equation (6) and corresponding equation in GTBP is, as one would expect, very small again. In all the places where m_0 appears in the (6), one should put $m_0 + m$ in order to obtain corresponding equation in GTBP. Equation for **r** (7) has again the same form as in GTBP. Right hand sides of equations (6) and (7) in this case could be treated as "perturbations" of two two-body problems (for **r** and **R**). Perurbations are then separated into two components in directions of vectors **R** and **r** and expressed in form of the series which should converge for sufficiently small Δ.

5. CONCLUSION

Differences and similarities of RTBP and GTBP are clearly showing that, by complexisity, RTBP stands somewhere between GTBP and restrected three body problems (previously formulated). On the other side Jacobi form of the equations of motion (equations (6) and (7)) give suitable form for step by step study of Hill's configuration of RTBP as well as GTBP (since they are very similar), by including higher order approximations. At the end we will note that planar version of this problem looks even more promising for further analysis.

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