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COMETARY TRAJECTORIES IN THE NONSTANDARD PLANE

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Abstract. Nonstandard (Leibnitz) analysis, based on nonstandard real numbers $^*\mathbf{R}$, introduces a specific mathematical method, as well as a way of thinking. It introduces *actual* infinitely small quantities and infinitely large quantities. Therefore, it gives good ground in considering physical systems which in idealized form have infinitely many degrees of freedom. Definitions and proofs are more intuitive, and its use is natural and intuitive whenever the considered physical system is composed of infinitely many particles. There are a lot of applications of nonstandard analysis based on this assumption in mathematical physics, in particular in quantum mechanics, fluid mechanics, dynamical systems, etc. Here, we discuss cometary trajectories, in particular the parabolic one, from the standpoint of nonstandard analysis. It appears that in a sense every parabola is an ellipse. Or from the standard point of view, every parabola P is a limit curve of a family *F* of confocal and coplanar ellipses. In a sense, the parabola P is also the envelope of the family *F*. Based on the observed data, this approach gives also a good mathematical model of paths of comets. In addition, this paper is written in order to promote the use of methods of nonstandard analysis in astronomy.

1. INTRODUCTION

The standard approach to physics is based on mathematics over $\mathbf{R}$, the field of real numbers. This structure is archimedean, i.e. it does not admit explicitely infinite quantities. We have no way of knowledge what a line in physical space is really like. It might be like the real line $\mathbf{R}$, the hyper-real line $^*\mathbf{R}$ which contains infinitesimals and infinite numbers, or neither. However, in applications of the mathematical analysis it is helpful to imagine a line in a physical space as $^*\mathbf{R}$. The hyper-real line is, like the real line, a useful mathematical model for a line in the physical space. One of the aims of this paper is to popularize the use of methods from nonstandard analysis (also known as Leibnitz analysis, non-archimedean analysis or Robinson's analysis) in studies of certain phenomena in astronomy. Here we shall discuss trajectories of comets from the stand point of Nonstandard analysis.

2. NONSTANDARD ANALYSIS

First, let us review the basic notions of nonstandard analysis. Newton and Leibnitz independently from each other developed differential calculus. By infinitesimals Leibnitz assumed "infinitely small numbers", and he performed the usual algebraic operations over them exactly in the same way as he did with real numbers. In particular, each positive infinitesimal ε in this contemplation was lesser than any ordinary real (standard) positive number, while $1/\varepsilon$ was greater than any standard positive number, i.e. $1/\varepsilon$ is an infinite number. The following rule was implicitly supposed:

Leibnitz principle: *Every mathematical proposition that is true for finite (real) numbers is also true for the extended system (i.e. system with infinite numbers), and vice versa.*

The major difficulty of Leibnitz's approach was a number of paradoxes and a lack of formal framework for consistent foundation of infinitesimal calculus. Introducing Weierstrass analysis the infinite quantities are expelled, for example the notion of the infinitesimal is replaced by the ε - δ formalism. In particular, zero-sequences (i.e. sequences are seen as infinitesimals. However, this is only an auxiliary notion there, and they lack the use of all algebraic operation (such as division) over them.

Abraham Robinson in Robinson (1961) solved the 300 years old problem of foundation of infinitesimal calculus. He founded Leibnitz analysis, i.e. introduced actual infinitely small and infinitely large numbers. They admit not only all algebraic operations, but also an application of usual functions from analysis (such as sin, cos, exp etc) over them. Robinson's solution was based on certain constructions and techniques from mathematical logic, such as the ultraproducts, the Compactness theorem and saturated models. The reader can find details about these notions in Chang and Keisler (1990).

The nonstandard analysis is based on properties of $^*\mathbf{R}$ and the transfer principle (Łoš theorem), the counterpart of the Leibnitz principle, which exchange propositions between $^*\mathbf{R}$ and $\mathbf{R}$. The nonstandard analysis has been used since then in explaining certain phenomena in physics, in particular in statistical physics and quantum mechanics (e.g. Anderson, 1976; Albeverio et al., 1986).

Mathematical models of nonstandard analysis are non-archimedean real fields enriched with nonstandard counterparts of notions of the mathematical analysis: elementary functions sin(x), ln(x), …, sets: natural numbers $\mathbf{N}$, integers $\mathbf{Z}$, rational numbers $\mathbf{Q}$, etc. As they are non-archimedean, they contain infinitesimals and infinite quantities. We can do the same constructions with more complex structures. We can build the nonstandard enlargement of any infinite structure: complex numbers **C**, the space of real sequences $\mathbf{R}^{\mathrm{N}}$, the space of real functions $\mathbf{R}^{\mathrm{R}}$, each having the metric on our choice; then infinite functional, geometrical and topological spaces. This construction simply allows us to do nonstandard but consistent mathematics. Leibnitz transfer principle enables one to translate theorems expressed by special, so called internal formulas from nonstandard universe to the standard one. Another useful property is expressed by the following theorem

Theorem (Extension property). Every function $f\colon \mathbf{R} \to \mathbf{R}$ can be extended to $f\colon {}^*\mathbf{R} \to {}^*\mathbf{R}$ which preserves all first order properties of f.

For example, if $f(x)= \sin(x)$, $g(x)= \cos(x)$, since $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$, the same identity holds for ${}^*f(x) = {}^*\sin(x)$ and ${}^*g(x) = {}^*\cos(x)$. Something similar is true for analytical continuations of real functions, but only for identities. In nonstandard analysis all first-order properties are preserved, including monotonicity, properties of zeros, etc. It is customary that the asterisk * is omitted in the case of elementary functions. So, $\sin(x)$ will denote ${}^*\sin(x)$ in ${}^*\mathbf{R}$ as well.

Another useful notion in nonstandard analysis is monads. An element $a \in {}^*\mathbf{R}$ is *finite* if there is positive integer n such that $-n < a < n$. By ${}^*\mathbf{R}_{\mathrm{fin}}$ we shall denote the set of all finite elements of ${}^*\mathbf{R}$. The *galaxy* of a is the set $\gamma(a)$ of nonstandard real numbers b such that $a - b$ is finite. In particular, ${}^*\mathbf{R}_{\mathrm{fin}} = \gamma(0)$. The mapping st: ${}^*\mathbf{R}_{\mathrm{fin}} \to \mathbf{R}$ (*standard part*) is defined by $\mathrm{st}(x)= \sup_{\mathbf{R}}\{y\colon y < x\}$. An *infinitesimal* is each finite ε such that $\mathrm{st}(\varepsilon) = \mathrm{st}(0) = 0$. The *monad* of 0 is the set $\mu(0)$ of all infinitesimals. Note that $\mu(0)$ is closed under addition and multiplication. Further, we say that numbers a and b are *infinitely close*, denoted by $a \approx b$, if $a - b \in \mu(0)$. In fact, $\mu(0)$ is the kernel of epimorphism st and it is a maximal ideal of the ring ${}^*\mathbf{R}_{\mathrm{fin}}$ The other monads we get by translations, i.e. $\mu(a)= a + \mu(0)$.

By use of homomorphism st one can replace the ε - δ formalism by algebraic identities. Let us illustrate this with several examples:

1. $f\colon \mathbf{R} \to \mathbf{R}$ is continuous iff (if and only if) for all $a \in {}^*\mathbf{R}_{\mathrm{fin}}$, $\mathrm{st}({}^*f(a))=f(\mathrm{st}(a))$.

2. Let $f\colon \mathbf{R} \to \mathbf{R}$ be a differentiable function and let $\varepsilon \neq 0$ be an infinitesimal. Then $f(x)' = \mathrm{st}({}^*f(x + \varepsilon) - f(x)/ \varepsilon)$. For example, $(x^2)'= \mathrm{st}(x^2 + 2x\varepsilon + \varepsilon^2-x^2)/ \varepsilon = \mathrm{st}(2x + \varepsilon) = 2x$.

3. If f is a continuous function, then the Riemann integral of f on the interval $[0,1]$ may be defined as $1/H\ \Sigma_{i=0}^{H}\ {}^*f(i/H)$, where H is an infinite number, i.e. $H \in {}^*\mathbf{N} \setminus \mathbf{N}$.

Detailed development of nonstandard analysis one can find in Stroyan and Luxemburg (1976).

Examples in Geometry and Astronomy.

As we saw, nonstandard analysis introduces *actual* infinitely small quantities and infinitely large quantities. Therefore, it gives good ground in considering physical systems which in idealized form have infinitely many degrees of freedom. Definitions and proofs are more intuitive, and it's use is natural and intuitive whenever the considered (idealized) physical system is composed of infinitely many particles. As an example, let us first consider Dirac delta function.

1. Dirac δ – function. Let $a(t)= \exp(-1/(1-|t|^2))$ if $|t| < 1$, $a(t)=0$ otherwise. This is a simple variation of Cauchy's flat function, and it belongs to the space E^∞ of infinitely many differentiable functions. Let ε be a positive infinitesimal, and let $b(t)= a(t/\varepsilon)$. Finally, let $k = \int_{-\infty}^{\infty} b(t)dt$ and let $\delta(t)= b(t)/k = a(t/\varepsilon)/k$. Then $\delta(t)$ belongs to ${}^*E^\infty$, it is positive, and has integral one. In fact, it is what is expected; $\delta(t)$ is a finite compact distribution and it has all properties attributed to the Dirac function.

2. **Tiling the Euclidean plane, Hao-Wang dominoes problem.** If there is a covering by the certain pattern of the finite type τ of each bounded domain in the plain such as squares and circles, prove that there is a cover of the type τ of the entire plane. One solution goes like this: by the Extension principle, we can find the covering C of the type $^*\tau$ of a square with edges having the infinite length H, i.e. $H \in {}^*\mathbf{N} \setminus \mathbf{N}$. Since τ is finite, we have $^*\tau = \tau$. Therefore, this particular nonstandard cover induces the covering of the entire Euclidean plane by restricting *C* to the standard (finite) part of ${}^*\mathbf{R} \times {}^*\mathbf{R}$.

In this example we have seen how to extend certain local property to the global one. We can try to interpret this covering property to the foundation of fundamental cosmological principles. Namely, all observations from the Earth are local, even in the large scale. But observations in the large scale show that the Universe is homogeneous and isotropic. Identifying observations with tiling, we see at once that we may assume two basic cosmological principles: homogeneity and isotropy of the Universe. Therefore, from the mathematical point of view at least it is consistent to assume so.

3. ELLIPSE IN THE NONSTANDARD PLANE

Let E be an ellipse having foci at the points (p,0) and (q,0) where p>0 is a positive real number and q>0 is an infinite real number. Then all standard points of E, i.e. the points lying in the real plane R^2, are the points of loci of an "ordinary" parabola P having the focus at (p,0). We show that P is the envelope of the family of all (standard) ellipses having one focus in (p,0) and the other one in (b,0), b is a positive real number.

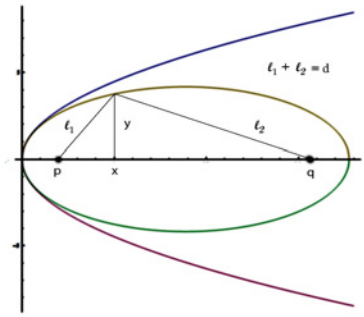


Figure 1: The ellipse having the foci in (p,0) and (q,0) with the vertex at the coordinate origin and $\ell_1+\ell_2=$ d, d=p+q, where ℓ_1 and ℓ_2 are distances of a point on the ellipse from the foci.

From the stated assumptions on the ellipse E, see Figure 1, we infer the following equations:

$$\begin{aligned} \ell_1 + \ell_2 &= d, \\ \ell_1^2 &= (x-p)^2 + y^2, \\ \ell_2^2 &= (x-q)^2 + y^2. \end{aligned} \tag{1}$$

By eliminating ℓ_1 and ℓ_2 from the set of formulas (1), we obtain the equation of the ellipse E:

$$y^2 = 4px - 4p(p(p + q)x + qx^2)/(p + q)^2 \tag{2}$$

We can interpret the formula (2) in the following two ways.

1. Ellipse in the nonstandard plane. Assume that $p \in$ R and that $x \in {}^*\mathbf{R}$ is finite and $q \in {}^*\mathbf{R}$ is infinite. Then the term

$$4p(p(p + q)x + qx^2)/(p + q)^2 \tag{3}$$

is an infinitesimal, while 4px is finite. So y is also finite and $y^2 \approx 4px$. Hence $st(y)^2 = 4p\ st(x)$, so by replacing st(x) by x and st(y) by y we obtain the equation $y^2 = 4px$ of parabola. Therefore, the standard part of the ellipse E in the nonstandard plane with the finite focus (p,0) and the infinite focus (q,0) is the parabola P determined by the equation $y^2 = 4px$. Observe that P does not depend on the choice of the infinite focus (q,0).

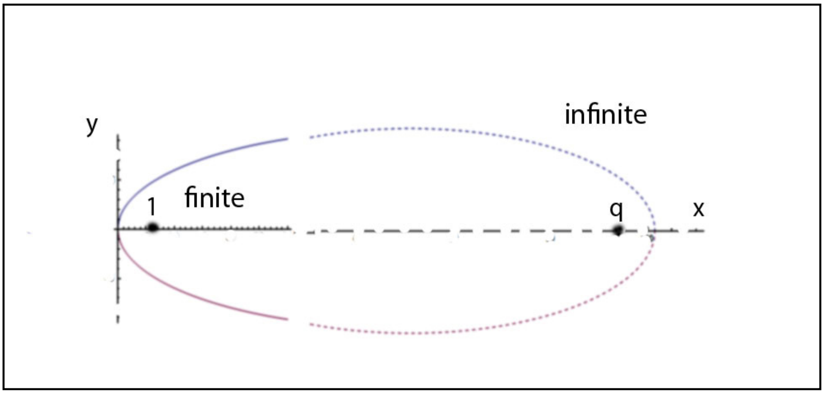


Figure 2: The ellipse E in the nonstandard plane.

It should be mentioned that all geometric and differential properties of the parabola P can be derived from the properties of the ellipse E. For example, the optical property that if a ray of light travels parallel to the symmetry axis of a parabola and strikes the concave side of the parabola, then it will be reflected to the focus follows immediately from the corresponding optical property of the ellipse E. Just note that if a ray r is coming from the (infinite) focus (q,0) it reflects from the ellipse to the focus (p,0) and that the standard part of r is a line parallel to the x-axis.

2. Family of confocal ellipses.

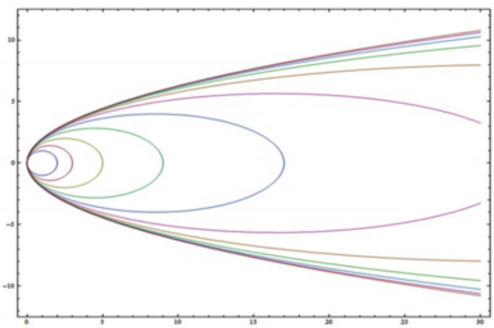


Figure 3: Family of confocal ellipses.

We may take (2) as the equation of the family of (standard) ellipses sharing the fixed focus (p,0), while the second focus (q,0) runs over the x-axis. Observe that from the astrodynamics point of view this family of ellipses may be regarded as Hohmann-Vetchinkin transfer orbits connecting co-planar circular orbits. We see that the parabola P is the limit curve enveloping ellipses from this family. However, it should be mentioned that P is not the envelope of the family of ellipses given by the equation (2) as it is defined in mathematical analysis. Namely, if a family of plane curves are given by a formula F(x,y,q) = 0, q is a parameter, then the mathematical envelope of this family is a curve touching each member of the family. The equation of the envelope is obtained by elimination of q from the system of equations $F(x,y,q)= 0$, $\partial F(x,y,q)/ \partial x = 0$. In our case, $F(x,y,q) = y^2 - 4px + 4p(p(p + q)x + qx^2)/(p + q)^2$, and it is easily found that the envelope is in fact the critical point $x = 0, y = 0$, the aphelia of q-ellipses.

4. COMETARY ORBITS

Most cometary orbits are very elongated. Namely, every cometary trajectory which is observed as parabolic actually is elliptical as further calculations show. But, the second focus is too remote to measure it. Many physical quantities related to the very elongated cometary orbits change for several orders of magnitude. For example, if the value of the velocity at the perihelion is assumed to be standard, then the velocity at aphelion may be taken as an infinitesimal. Therefore, we shall consider cometary trajectories assuming nonstandard analysis. By our consideration in the previous section we may assume that every parabolic trajectory is an ellipse. Our discussion is relied on available cometary data, so we shall first shortly review them.

The number of observed comets is rapidly growing due to the development of space technology. For example the ESA/NASA SOHO spacecraft, http://www.nascom.nasa.gov, discovered exactly 1500 comets since 1995, the last one on 27. June 2008. About 2300 are catalogued, even if it is believed that there are more than 10^9 of them. As very few comets have periods of 12 years, their trajectories are good illustration for very elongated or nearly parabolic ellipses. Here is the short history on recent comet discoveries.

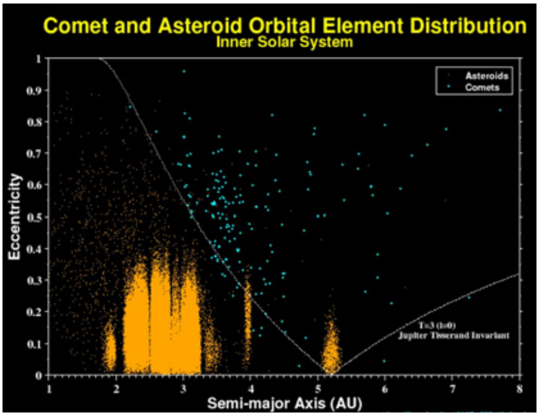
The Catalog of Cometary Orbits, compiled by Marsden, 1989 edition, lists 1292 computed orbits from 239 BC to AD 1989; only 91 of them were computed using the rare accurate historical datafrom before the 17th century. More than 1200 are therefore derived from cometary passages during the last three centuries. Sets of orbital elements in Marsden's catalog involve only 810 individual comets; the remainder represents the repeated returns of periodic comets. Four of these comets had been definitely lost, and three more were probably lost, presumably because of their decay in the solar heat. Of the 155 short-period comets, 93 have been observed at two or more perihelion passages.

The 16th edition of the Catalogue of Cometary Orbits of Smithsonian Astrophysical Observatory issued in 2005 contains 3031 sets of orbital elements (in the J2000.0 system) for 2991 cometary emersions of 2221 different comets through mid-August 2005. There is a special tabulation giving osculating elements for the 170 numbered periodic comets, excluding seven deemed to be lost.

According to the list of periodic comets on the Planetary Data System Small Bodies Node, NASA, last update on 10 April 2008, http://pdssbn.astro.umd.edu/comet\data, there are 420 designated periodic comets. According to Seiichi Yoshida's Comet Catalog, www.aerith.net, there are 243 non-numbered periodic comets (the last discovered C/2008 L3, 13. Jun 2008) and 200 numbered periodic comets.

In next discussion we shall rely on Marsden Catalog of Cometary Orbits. Of the 655 comets of long period contained in the Catalog, 192 have osculating elliptic orbits, and 122 have osculating orbits that are very slightly hyperbolic. Finally, 341 are listed as having parabolic orbits, but this is rather false because either it has not been possible to detect unequivocal deviations from a parabola on the usually very short arc along which the comets have been observed or, more simply, the final calculations have never been made. However, the parabola is always assumed first in the preliminary computation. If the osculating orbit is computed backward to when the comet was still far beyond the orbit of Neptune and if the orbit is then referred to the centre of mass of the solar system, the original orbits almost always prove to be elliptic.

These data exactly validates methods of nonstandard analysis in studying cometary trajectories. For example, in the preliminary computation, the value of the parameter p is computed. Simply, the second term (3) in the formula (2) may be omitted as we may consider it as an infinitesimal. It also shows that the formula (2) could be very appropriate in calculation of cometary orbits.



Let us consider very-long-period comets and comets having orbits not significantly different from a parabola. It is believed that these comets originate in the Oort cloud which is distant around 100000 AU from the Sun. By our previous discussion it is appropriate to use here methods of nonstandard analysis. So let us assume that a hypothetical comet C is moving along an ellipse E in the nonstandard plane having the second focus at $(q,0)$ where q is an infinite number. Therefore, the aphelion of E is at infinity, and by the second Kepler's law the velocity v of the comet near the aphelia (i.e. at the finite distance from aphelia in terms of nonstandard analysis) is an infinitesimal. Otherwise, the surface swept by the comet for the finite time Δt would be infinite due to the infinite distance of the comet from the Sun, and that would contradict the Second Kepler's law. In reality, a simple calculation shows that the velocity v of the comet C near aphelia would be around 100 m/sec, negligible small comparing to the velocity at the perihelia. Therefore, the momentum $p = mv$ of the comet C is an infinitesimal too; we would rather say that the comet C floats in the Oort cloud instead of it travels around the Sun. Hence the trajectory of the comet C is subject to any small perturbation, i.e. any infinitely small force, or impulse, would change significantly its trajectory. Simply saying, *parabolic orbits at far distances are very unstable*. This follows from the fact that the velocity Δv needed for transition from an orbit O_2 to the transfer ellipse which would carry the comet C to the other orbit O_1 is an infinitesimal. The transfer ellipse will be seen from the near neighborhood of the Sun as a parabola with the second focus at infinity. The graph above (A. Chamberlin, JPL/Caltech, 2007) illustrates the instability of cometary orbits. The dotted line represents the Jupiter Tisserand invariant (T) evaluated at T=3 and zero inclination. This boundary *very roughly* separates small-bodies which are dynamically bound to Jupiter from those which are not. The region above this curve represents objects with T<3 (i.e. bound to Jupiter). Notice that *most* comets as well as the Trojan asteroids appear in that upper region (T<3, bound to Jupiter) while *nearly* all asteroids are contained in the region below the curve where T>3 (i.e. not bound to Jupiter). Therefore most of the comets having now elliptical orbits were captured once in the past by Jupiter, the dominant planet of the Solar system.

There are other astronomical evidences that support our discussion. Namely, according to Delsemme (2008) among the very-long-period comets, there is a particular class of comets that Oort showed as having never passed through the planetary system before, notwithstanding the fact that their original orbits were elliptic, which implies repeated passages. This paradox vanishes when it is understood that their perihelia were outside of the planetary system before their first appearance but that their orbits have been perturbed near aphelia by interstellar-cloud passages or by galactic tides, in such a way that their perihelia were lowered into the planetary system.

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